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Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications

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Abstract

We perform a Hodge theoretic study of parameter dependent families of D-branes on compact Calabi–Yau manifolds in type II and F-theory compactifications. Starting from a geometric Gauss-Manin connection for B-type branes we study the integrability and flatness conditions. The B-model geometry defines an interesting ring structure of operators. For the mirror A-model this indicates the existence of an open-string extension of the so-called A-model connection, whereas the discovered ring structure should be part of the open-string A-model quantum cohomology. We obtain predictions for genuine Ooguri-Vafa invariants for Lagrangian branes on the quintic in $\mathbf{P}^4$ that pass some non-trivial consistency checks. We discuss the lift of the brane compactifications to F-theory on Calabi–Yau four-folds and the effective couplings in the effective supergravity action as determined by the $\mathcal{N}=1$ special geometry of the open-closed deformation space.

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1. Introduction

Much of the success of closed string mirror symmetry relies on the identification of deformation families of topological A- and B-models obtained by perturbing a reference theory by marginal operators. An important application is the construction of the mirror map and the so-called A-model connection from the flat sections of the Gauss-Manin connection in the B-model. Given the understanding of mirror symmetry as an equivalence of D-brane categories [1] it seems natural to contemplate on an open string version of mirror symmetry which identifies families of open topological A- and B-models perturbed by both open and closed string operators.

There is an immediate puzzle, however, as there is a superpotential in the open-string sector computed by a Chern-Simons functional [2]. A non-trivial extension of the mirror map to the open string deformation space therefore appears to require some sort of off-shell version of mirror symmetry. This somewhat worrisome issue has been successfully circumvented for non-compact toric branes as defined in [3], which led to many important developments in open topological strings and matrix models, and for compact, but apparently rigid branes, which have been recently considered in a series of works following ref. [4].

Nevertheless it could be rewarding to pursue the idea of a version of mirror symmetry that identifies families of A- and B-models perturbed by closed and open string deformations. Such a map would likely involve interesting mathematical structures, such as a non-trivial open-closed chiral ring and an open-string version of quantum cohomology as its geometrical A-model counterpart. Indeed, open-string deformations are important in the definition of enumerative Ooguri-Vafa invariants [5] for world-sheets with boundaries, and it seems natural to ask about an A-model quantum product specified by these open-string invariants.

In this note we approach the problem from a Hodge theoretic point of view, using a certain relative cohomology space introduced in refs. [6,7] as a geometric model for the open-closed deformation space of the B-model. Starting from the Gauss-Manin connection on this space, we derive some predictions on an open string extension of the so-called A-model connection and the associated invariants, an interesting quantum ring structure of operators and a conjectural metric on the open-closed deformation space. We obtain explicit results for an example of a family of branes on the quintic in $\mathbf{P}^4$, which satisfy some non-trivial consistency checks. We stress already at this point that our arguments rely mostly on the geometric Hodge structure for the B-model and we do not have a proper understanding of the A-model side and a world-sheet derivation of these results

at the moment. This would involve, amongst others, a better understanding of the (off-shell) bulk-boundary ring and its realization in the A-model. Instead we work out some predictions on these structures from the geometric ansatz on the B-model side.

The open-closed deformation space $\mathcal{M}$ defined in this way is not a generic Kähler manifold as allowed by the effective $\mathcal{N}=1$ supergravity, but of a restricted type controlled by the $\mathcal{N}=1$ special geometry of ref. [6]. This can be explained by the fact that although the derivations in this note start from mirror pairs of branes on Calabi–Yau three-folds, there is a close connection to Calabi–Yau four-fold compactifications of a type discussed in [8,9]. As will be argued below, the (B-type version) of the four-fold compactification defines an F-theory embedding for the compactification of the B-type branes on the four-fold Z^*. A crucial point is the existence of a weak (de-)coupling limit $g_s \to 0$ in the B-brane geometry, which we identify as the mirror of a "large base" limit of the fibration in the mirror four-fold. Amongst others, the F-theory dual determines the superpotential and the Kähler potential of the $\mathcal{N}=1$ effective supergravity theory as well as subleading corrections in the string coupling g_s, which should be relevant for phenomenological issues.

The organization of this note is as follows. In sect. 2 we introduce the deformation space $\mathcal{M}$ of a certain family of B-branes on compact Calabi-Yau three-folds. In sect. 3 we derive a generalized hypergeometric system for the Hodge variation on the relevant cohomology space. In sect. 4 we study the flatness and integrability conditions of the Gauss-Manin connection and define the mirror map on the open-closed deformation space. Sect. 5 contains a case study for a family of branes on the quintic and a prediction for Ooguri–Vafa invariants. In sect. 6 we make some remarks from the world-sheet and CFT point of view together with a discussion on the open-closed chiral ring. In sect. 7 we discuss the effective $\mathcal{N}=1$ supergravity action for the type IIB/F-theory compactification and identify the weak coupling limit. We determine the superpotential and propose a Kähler metric on the open-closed deformation space $\mathcal{M}$; some details of the necessary computations are relegated to app. A. Sect. 8 contains a brief summary of the results and our conclusions.

2. Geometry and deformation space of the B-model

We start with the definition of the geometrical structure that will be taken as a model for the open-closed deformation space $\mathcal{M}$, following refs. [6,7,9]. Let (Z, Z^*) be a mirror pair of Calabi–Yau three-folds and (L, E) a mirror pair of A/B-type branes on it. On-shell, the classical A-type brane geometry is perturbatively defined by a special

Lagrangian submanifold $L \in H_3(Z)$ together with a flat bundle on it [2]. At the quantum level non-perturbative open worldsheet instantons may couple to the special Lagrangian submanifold L. Then an on-shell quantum A-type brane arises if the classical geometry is not destabilized by such instanton corrections [10]. The mirror B-type geometry consists of a holomorphic sheaf E on Z^* describing a D-brane with holomorphic gauge bundle wrapped on an even-dimensional cycle. The concrete realization and application of open string mirror symmetry to this brane geometry, which will be central to all of the following, has been formulated in the pivotal work [3]. More details on the action of mirror symmetry on brane geometries can be found in refs. [11,12].

The moduli space of the closed string B-model on Z^* is the space $\mathcal{M}_{CS}$ of complex structures, parametrizing the family $\mathcal{Z}^* \to \mathcal{M}_{CS}$ of 3-folds with fiber $Z^*(z)$ at $z \in \mathcal{M}_{CS}$. Here $z = \{z_a\}$, $a = 1, ..., h^{2,1}(Z^*)$ denote some local coordinates on $\mathcal{M}_{CS}$. An important concept in the Hodge theoretic approach to open string mirror symmetry of refs. [6,7,9] is the definition of an off-shell deformation space $\mathcal{M}$, which includes open string deformations. To study the obstruction superpotential on $\mathcal{M}$, one first defines $\mathcal{M}$ as an *unobstructed* deformation space for a relative homology problem and studies the functions $\underline{\Pi}^\Sigma : \mathcal{M} \to \mathbf{C}$ defined by integration over the dual cohomology space. In a second step, one adds an obstruction, which can be shown to induce a superpotential on $\mathcal{M}$ proportional to a linear combination of these 'relative periods' $\underline{\Pi}^\Sigma$.

The unobstructed moduli space for the family of relative cohomology groups can be defined as the moduli space of a *holomorphic* family of hypersurfaces embedded into the family $\mathcal{Z}^*$ of CY 3-folds [6,7]

$$\begin{aligned} i: \qquad & \mathcal{D} \hookrightarrow \mathcal{Z}^* \\ & D(z,\hat{z}) \hookrightarrow Z^*(z) \end{aligned} \tag{2.1}$$

where[1] $\hat{z} = \{\hat{z}_\alpha\}$ are local coordinates on the moduli space of the embeddings $i : D(z, \hat{z}) \hookrightarrow Z^*(z)$ for fixed complex structure z. The total moduli space $\mathcal{M}$ of this family is the fibration

$$\begin{tikzcd} \hat{\mathcal{M}}(\hat{z}) \arrow[r] & \mathcal{M} \arrow[d, "\pi"] \\ & \mathcal{M}_{CS}(z) \end{tikzcd} \tag{2.2}$$

[1] Here and in the following we often use a hat to distinguish data associated with the open string sector.

where the point $z \in \mathcal{M}_{CS}$ on the base specifies the complex structure on the CY 3-fold $Z^*(z)$ and the point $\hat{z} \in \hat{\mathcal{M}}$ on the fiber the embedding. In the context of string theory, the moduli z and $\hat{z}$ arise from states in the closed and open string sector, respectively. Note that the fields associated with the fiber and the base of $\mathcal{M}$ couple at a different order in string perturbation theory. This will be relevant when defining a metric on $T\mathcal{M}$ in sect. 7.

Following [6,13,7,9], we consider functions on the unobstructed deformation space $\mathcal{M}$ given by 'period integrals' on the relative cohomology group defined by the brane geometry. The embedding $i : D \hookrightarrow Z^*$ defines the space $\Omega^*(Z^*, D)$ of relative p-forms via the exact sequence

$$0 \longrightarrow \Omega^*(Z^*, D) \longrightarrow \Omega^*(Z^*) \xrightarrow{i^*} \Omega^*(D) \longrightarrow 0 \ .$$

The associated long exact sequence defines the relative three-form cohomology group

$$H^3(Z^*, D) \ \simeq \ \ker \big(H^3(Z^*) \to H^3(D)\big) \oplus \mathrm{coker} \ \big(H^2(Z^*) \to H^2(D)\big) \ , \qquad (2.3)$$

which provides the geometric model for the space of groundstates of the open-closed topological B-model. In a generic[2] situation, the first summand equals $H^3(Z^*)$ and represents the closed string sector capturing the deformations of the complex structure of Z^*. The relation of the above sheaf cohomology groups considered in [6,7] and the Ext groups studied in ref. [14] will be discussed in sect. 6.

By eq.(2.3), a closed relative three-form $\underline{\Phi} \in \Omega^3(Z^*, D)$, representing an element of $H^3(Z^*, D)$, can be described by a pair (Φ, ϕ), where Φ is a 3-form on Z^* and ϕ a 2-form on D. The differential is $d\underline{\Phi} = (d\Phi, i^*\Phi - d\phi)$ and the equivalence relation $(\Phi, \phi) \sim (\Phi, \phi) + (d\alpha, i^*\alpha - d\beta)$ for $\alpha \in \Omega^2(Z^*)$, $\beta \in \Omega^1(D)$. The duality pairing between a 3-chain class $\gamma_\Sigma \in H_3(Z^*, D)$ and a relative p-form class $[\underline{\Phi}]$ is given by the integral

$$\int_{\gamma_\Sigma} \underline{\Phi} = \int_{int(\gamma_\Sigma)} \Phi - \int_{\partial\gamma_\Sigma} \phi \ . \qquad (2.4)$$

The fundamental holomorphic objects of the open-closed topological B-model are particular examples of eq.(2.4), namely the relative period integrals of the holomorphic $(3,0)$ form $\underline{\Omega}$ on Z^*, viewed as the element $(\Omega, 0) \in H^3(Z^*, D)$, over a basis $\{\gamma_\Sigma\}$ of topological 3-chains:

$$\underline{\Pi}^\Sigma(z, \hat{z}) = \int_{\gamma_\Sigma} \underline{\Omega}, \qquad \gamma_\Sigma \in H_3(Z^*, D) \ . \qquad (2.5)$$

[2] That is $H^1(Z^*) \simeq 0$ and we made the simplifying assumption that D is ample, which is a reasonable condition on the divisor wrapped by a B-type brane. The Lefschetz hyperplane theorem then implies $H^1(D) \simeq 0$ and, by Poincaré duality, $H^3(D) \simeq 0$.

The cohomology group $H^3(Z^*, D)$ is constant over $\mathcal{M}$, but the Hodge decomposition $F^pH^3(Z^*, D)$ and the direction of the (3,0) form $\underline{\Omega}$ varies with the moduli. The period integrals $\underline{\Pi}^{\Sigma}(z, \hat{z})$ thus define a set of moduli dependent local functions on $\mathcal{M}$. Despite the fact, that there is not yet a superpotential on $\mathcal{M}$, these functions should have an important physical meaning in the unobstructed theory as well. In sect. 7 we argue that they define a Kähler metric on $\mathcal{M}$ and thus determine the kinetic terms of the bulk and brane moduli in the effective action.

Further details on the relation between relative cohomology and open-closed deformation spaces can be found in refs. [6,15,13,7,16,17]. For the mathematical background, see e.g. refs. [18,19].[3]

Obstructed deformation problem

The physical meaning of the period integrals is altered after adding an additional lower-dimensional brane charge on a 2-cycle, which induces an obstruction on $\mathcal{M}$. From a physics point of view this perturbation may be realized by either adding an additional brane on a 2-cycle in D or by switching on a 2-form gauge flux on the original brane on D. A world-sheet derivation of the obstruction from the relevant Ext groups in the open string CFT will be given in sect. 6.

In the Hodge theoretic approach of refs. [6,7,9], the superpotential on $\mathcal{M}$ in the obstructed theory is given by a certain linear combination of the relative periods (2.5) of the unobstructed theory, as reviewed below. This is similar to the case of closed string flux compactifications, where the flux superpotential on the space $\mathcal{M}_{CS}$ of complex structures can be computed in the unobstructed theory with $\mathcal{M}_{CS}$ as a true moduli space [21,22,23].

Let C_i denote the irreducible components of the 2-cycle carrying the additional brane charge and $C = \sum_i C_i$ their sum. If $[C] = 0$ as a class in $H_2(Z^*)$, there exists a 3-chain Γ in the sheaf cohomology group (2.3), with $\partial\Gamma = C$. In particular, the choice of the brane cycle C restricts the relevant co-homology to the subspace

$$H_3(Z^*, D) \ \longrightarrow \ H_3(Z^*, \sum_i C_i) \ . \tag{2.6}$$

The open-closed string superpotential $W(z, \hat{z})$ on $\mathcal{M}$ for this brane configuration is computed by a relative period integral $\underline{\Pi}(z, \hat{z})$ on this subspace [6,7,9].

[3] After publication of this work, a thorough mathematical justification and generalization of the methods of refs.[6,7,9] appeared in ref. [20], with some overlap with sect. 3 below.

It was argued in [24], that a superpotential, that has the correct critical points to describe a supersymmetric brane on C, is given by the chain integral

$$\mathcal{T} = \int_{\gamma(C)} \Omega \qquad \partial\gamma(C) \neq 0 \ . \tag{2.7}$$

This expression was later derived from a dimensional reduction of the holomorphic Chern-Simons functional of ref. [2] in refs. [10,3].[4]

As it stands, eq.(2.7) can be viewed either as a definition in absolute cohomology, or in relative cohomology, replacing $\Omega \to (\Omega, 0)$ and including the explicit boundary term in eq.(2.5). The difference is important only off-shell and in this way the relative cohomology ansatz of refs.[6,7,9], building on the results of [3], can be viewed as a particular proposal for an off-shell definition of the superpotential.

In absolute cohomology, the integral (2.7) is a priori ill-defined because of non-vanishing boundary contributions from exact forms, which do not respect the equivalence relation $[\Omega] = [\Omega + d\omega]$. To obtain a well-defined pairing one may restrict homology to chains with boundary $\partial\gamma$ a holomorphic curve and cohomology to sections of the Hodge subspace $F^2H^3 = H^{3,0} \oplus H^{2,1}$ [19].[5] This is the normal function point of view taken in refs. [4,25]. Since the curve $C = \partial\gamma$ being holomorphic corresponds to a critical point $dW = 0$ of the superpotential with respect to the open string moduli [24], continuous open-string deformations are excluded from the beginning and one obtains the critical value $W^{crit}(z)$ of the superpotential as a function of the closed-string deformations z, only. The dependence of the critical superpotential $W_{crit}(z)$ on the closed string moduli z is still a highly interesting quantity and at the center of the works [4,25] on open string mirror symmetry, which gave the first computation of disc instantons in compact CY 3-fold from mirror symmetry. The dependence of the superpotential on open string deformations $\hat{z}$ is not captured by this definition.

In the relative cohomology ansatz of refs. [6,7,9], the pairing (2.7) is well-defined in co-homology also away from the critical points as a consequence of enlarging the co-homology spaces as in (2.3). The extra contribution to $H^3(Z^*, D)$ from the second factor in (2.3) describe additional degrees of freedom in the brane sector. According to this proposal, the relative periods $\underline{\Pi}(z, \hat{z})$ on the subspace $H^3(Z^*, C)$ describe the 'off-shell' superpotential

[4] More precisely, the chain integral gives the tension $\mathcal{T}$ of a domain wall realized by a brane wrapped on the 3-chain $\gamma(C)$.

[5] The potentially ambiguous boundary terms then vanish as $\int_\gamma \Omega + d\omega = \int_\gamma \Omega + \int_{\partial\gamma} \omega = \int_\gamma \Omega$ for ω a (2,0) form and $\partial\gamma$ a 2-cycle of type $(1,1)$.

$W(z,\hat{z})$ depending on brane deformations $\hat{z}$. For consistency, $W(z,\hat{z})$ should reduce to the critical superpotential $W^{crit}(z)$ at the critical points. This has been verified for particular examples in refs.[7,9].

Although we eventually end up with studying the periods on the restricted subspace $H^3(Z^*,C)$ in (2.6) for a fixed brane charge C, the introduction of the larger relative cohomology space $H_3(Z^*,D)$ was not redundant, even for fixed choice of obstruction brane C, as it was crucial for the definition of the finite-dimensional off-shell deformation space $\mathcal{M}$, on which the obstruction superpotential can be defined. The off-shell deformation space for a brane on C is generically infinite-dimensional, with most of the deformations representing heavy fields in space-time that should be integrated out. To define an effective superpotential we have to pick an appropriate set of 'light' fields and integrate out infinitely many others.

The ansatz of refs. [6,7,9] to define $\mathcal{M}$ by perturbing the unobstructed moduli space of a family $\mathcal{D}$ of hypersurfaces is thus not a circuitry, but rather a systematic way to define a finite-dimensional deformation space with parametric small obstruction, together with a local coordinate patch, on which a meaningful off-shell superpotential can be defined. As C can be embedded in different families of hyperplanes, the parametrization of the deformation space depends on the choice of the family $\mathcal{D}$ and this corresponds to a different choice of light fields for the effective superpotential.[6] Each choice covers only a certain patch of the off-shell deformation space and there will be many choices to parametrize the same physics and mathematics near a critical locus by slightly different relative cohomology groups. This choice of a set of light fields is inherent to the use of effective actions and should not be confused with an ambiguity in the definition.[7]

In the context of open string mirror symmetry, the most interesting aspect of the deformation spaces $\mathcal{M}$ constructed in this way is the presence of 'almost flat' directions in the open string sector, which lead to the characteristic A-model instanton expansion of the superpotential, as will be shown in in sect. 5. The result passes some non-trivial consistency checks which provides some evidence in favor of this definition of off-shell string mirror symmetry.[8] On the other hand, for general massive deformations, one would not

[6] One could always combine these 'different' families into a single larger family at the cost of increasing the dimension of the deformation space $\mathcal{M}$.

[7] An attempt to reformulate the relative cohomology approach of refs. [6,7,9] by using the excision theorem, as contemplated on in ref. [17], is thus likely to produce just another parametrization corresponding to a slightly different choice of light fields, rather then a distinct description.

[8] See also refs. [9,26] for additional examples and arguments.

expect the simple notions of flatness and an integral instanton expansion observed in this paper.

The most general superpotential captured by the relative cohomology group $H^3(Z^*,C)$ includes also a non-trivial closed string flux on $H^3(Z^*)$ and the two contributions can be combined in the general linear combination of relative period integrals [6,7]

$$W_{\mathcal{N}=1}(z,\hat{z}) = \sum_{\gamma_\Sigma \in H^3(Z^*,D)} \underline{N_\Sigma}\, \underline{\Pi}^\Sigma(z,\hat{z}) \ = W_{closed}(z) + W_{open}(z,\hat{z}), \tag{2.8}$$

where the contributions from the closed and open string sector are, with $\underline{N_\Sigma} := (N_\Sigma, \hat{N}_\Sigma)$,

$$W_{closed}(z) = \sum_{\gamma_\Sigma, \partial\gamma_\Sigma = 0} N_\Sigma\, \underline{\Pi}^\Sigma(z), \qquad W_{open}(z,\hat{z}) = \sum_{\gamma_\Sigma, \partial\gamma_\Sigma \neq 0} \hat{N}_\Sigma\, \underline{\Pi}^\Sigma(z)\ .$$

This is the superpotential for a four-dimensional $\mathcal{N}=1$ supersymmetric string compactification with N_Σ and $\hat{N}_\Sigma$ quanta of background "fluxes" in the closed and open string sector, respectively. The first term $W_{closed}(z)$ is proportional to the periods over *cycles* $\gamma_\Sigma \in H^3(Z^*)$ and represents the closed string "flux" superpotential for N_Σ "flux" quanta [27,22,23]. The second term captures the chain integrals (2.7). Note that there are contributions to the superpotential from different orders in the string coupling and the instanton expansion of the mirror A-model will involve *sphere and disc instantons* at the same time.[9]

There are two important points missing in the above discussion, which will be further studied in the following. One is the selection of the proper homology element $\gamma(C)$ that computes the superpotential, given a 2-cycle C representing the lower-dimensional brane charge. The other one is the mirror map, which allows to extract a prediction for the disc and sphere instanton expansion for the A-model, starting from the result obtained from the relative periods of the B-model. The additional information needed to answer these questions comes from the variation of mixed Hodge structure on the Hodge bundle with fiber the relative cohomology group $H^3(Z^*,D)$. The Hodge filtration defines a grading by Hodge degree p of the cohomology space at each point $(z,\hat{z})$. In closed string mirror symmetry, restricting to $H^3(Z^*) \subset H^3(Z^*,D)$, this grading is identified with the $U(1)$ charge of the chiral ring elements in the conformal field theory on the string world-sheet. A similar interpretation in terms of an open-closed chiral ring has been proposed in refs. [6,7]. The upshot of this extra structure is, that there are *two* relevant relative period integrals

[9] This links to the open-closed string duality to Calabi–Yau 4-folds of refs. [8,9]; see also [28] for a recent discussion.

associated with the brane charge C, distinguished by the grading, such that one gives the mirror map to the A-model, and the other one the superpotential [6,7,9].

In the following sections we thus turn to a detailed study of the variation of the mixed Hodge structure on the relative cohomology group $H^3(Z^*, D)$, which we take as a geometric model for the variation of the states of the open-closed B-model over the deformation space $\mathcal{M}$. In sect. 3 we derive a set of differential equations, whose solutions determine a basis for the periods $\underline{\Pi}^{\Sigma}(z, \hat{z})$ on $\mathcal{M}$ in terms of generalized hypergeometric functions. In sect. 4 we study the mixed Hodge variation on the relative cohomology bundle, which leads to the selection of the proper functions for the mirror map and the superpotential.

3. Generalized hypergeometric systems for the B-model

In the first step we derive a generalized hypergeometric system of differential operators for the deformation problem defined above, in the concrete framework of toric branes on toric CY hypersurfaces defined in ref. [3] and further scrutinized in [9,26]. The result is a system of differential equations acting on the relative cohomology space and its periods, whose associated Gauss-Manin system and solutions will be studied in the next section. The result is summarized in eq. (3.21); the reader who is not interested in the derivation may safely skip this section.

To avoid lengthy repetitions, we refer to refs. [3,9] for the definitions of the family of toric branes in compact toric hypersurfaces, to refs. [29,12] for background material on mirror symmetry and toric geometry and to refs. [30,31,32] for generalized hypergeometric systems for the closed-string case. The notation is as follows : Δ is a reflexive polyhedron in $\mathbf{R}^5$ defined as the convex hull of p integral vertices $\nu_i \subset \mathbf{Z}^5 \subset \mathbf{R}^5$ lying in a hyperplane of distance one to the origin.[10] $W = P_{\Sigma(\Delta)}$ is the toric variety with fan $\Sigma(\Delta)$ defined by the set of cones over the faces of Δ. Δ^* is the dual polyhedron and W^* the toric variety obtained from $\Sigma(\Delta^*)$. The mirror pair of toric hypersurfaces in (W, W^*) is denoted by (Z, Z^*).

[10] We use the standard convention, identify the interior point ν_0 of Δ with the origin, and specify the vertices by four components $\nu_{i,k},\ k = 1, ..., 4$, i.e. $\nu_0 = (0, 0, 0, 0)$; see refs. [32,31,33] for more details.

3.1. Generalized Hypergeometric systems for Calabi-Yau three-folds

The p (relevant) integral points of Δ determine the hypersurface $Z^* \subset W^*$ as the vanishing locus of the equation

$$P(Z^*) = \sum_{i=0}^{p-1} a_i y_i = \sum_{\nu_i \in \Delta} a_i X^{\nu_i}$$

where a_i are complex parameters, y_i are certain homogeneous coordinates [34] on W^*, X_k, $k = 1, ..., 4$ are inhomogeneous coordinates on an open torus $(\mathbf{C}^*)^4 \subset W^*$ and $X^{\nu_i} := \prod_k X_k^{\nu_{i,k}}$ [32]. The integral points ν_i and the homogeneous coordinates y_i fulfill $h^{1,1}(Z) = h^{2,1}(Z^*)$ relations

$$\sum_{i=0}^{p-1} l_i^a \nu_i = 0 \ , \qquad \prod_{i=0}^{p-1} y_i^{l_i^a} = 1 \ , \qquad a = 1, ..., h^{1,1}(Z). \tag{3.1}$$

The p-dimensional integral vectors l^a specify the charges of the matter fields of the gauged linear sigma model (GLSM) associated with Z [35]. The index 0 refers to the special field p of negative charge which enters linearly in the two-dimensional GLSM superpotential.

An important datum for the B-model on the mirror manifold Z^* are the period integrals of the holomorphic (3,0) form Ω. The fundamental period integral on Z^* can be defined as [30]

$$\Pi(a_i) = \frac{1}{(2\pi i)^4} \int_{|X_k|=1} \frac{1}{P(Z^*)} \prod_{k=1}^{4} \frac{dX_k}{X_k} \ . \tag{3.2}$$

As noted in refs. [30,32], the period integral is annihilated by a system of differential operators of the so-called GKZ hypergeometric type [36]

$$\begin{aligned} \mathcal{L}(l) &= \prod_{l_i>0} (\frac{\partial}{\partial a_i})^{l_i} - \prod_{l_i<0} (\frac{\partial}{\partial a_i})^{-l_i}, \qquad l \in K, \\ \mathcal{Z}_k &= \sum_{i=0}^{p-1} \nu_{i,k} \vartheta_i \ , k = 1, ..., 4; \qquad \mathcal{Z}_0 = \sum_{i=0}^{p-1} \vartheta_i + 1 \ . \end{aligned} \tag{3.3}$$

where $\vartheta_i = a_i \partial_{a_i}$ and K denotes the set of integral linear combinations of the charge vectors l^a. The differential equations $\mathcal{L}(l)\,\Pi(a_i) = 0$ follow straightforwardly from the definition (3.2). The equations $\mathcal{Z}_k\,\Pi(a_i) = 0$ express the invariance of the period integral under the torus action and imply that the periods depend, up to normalization, only on special combinations of the parameters a_i, $\Pi(a_i) \sim \Pi(z_a)$, where

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a} \ , \tag{3.4}$$

define $h^{2,1}(Z^*) = h^{1,1}(Z)$ local coordinates on the complex structure moduli space of Z^* [31].

3.2. Extended hypergeometric systems for relative periods

We proceed with the derivation of a GKZ hypergeometric system which annihilates the relative period integrals (2.5) on the relative cohomology group. The definition of the (union of) hypersurfaces D cannot preserve all torus symmetries. Instead (some of) the torus actions move the position of the branes.[11] As a consequence, the relative periods are no longer annihilated by all the operators $\mathcal{Z}_k$ and depend on additional parameters specifying the geometry of D. The differential equations (3.3) for the period integrals imply on the level of forms

$$\mathcal{L}(l)\ \Omega = d\omega(l)\ ,$$
$$\mathcal{Z}_k\ \Omega = d\omega_k\ .$$

The exact terms on the r.h.s. contribute only to integrals over 3-chains $\hat{\gamma} \in H_3(Z^*, D)$ with non-trivial boundaries $\partial\hat{\gamma}$. The modification of the differential equations for the relative periods can be computed from these boundary terms.

To keep the discussion simple we derive the differential operators for the relative periods on the mirror of the quintic and present general formulae at the end of this section. The integral points of the polyhedron Δ are

$$\begin{aligned} &\nu_0 = (0,0,0,0)\ , \quad \nu_1 = (1,0,0,0)\ , \quad \nu_2 = (0,1,0,0)\ , \quad \nu_3 = (0,0,1,0)\ , \\ &\nu_4 = (0,0,0,1)\ , \quad \nu_5 = (-1,-1,-1,-1)\ , \end{aligned} \tag{3.5}$$

leading to the defining polynomial for the mirror quintic:

$$P(Z^*) = a_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5(X_1X_2X_3X_4)^{-1}\ .$$

The holomorphic (3,0) form on the mirror quintic can be explicitly represented as the residuum

$$\Omega = \mathrm{Res}_P\ \frac{a_0}{P}\prod_k \frac{dX_k}{X_k}\ , \tag{3.6}$$

at $P = 0$ [30]. Here we have changed the normalization with respect to eq. (3.2) to the standard convention $\Pi(a_i) \to a_0\Pi(a_i)$. The GLSM for the quintic is specified by one charge vector $l^1 = (-5,1,1,1,1,1)$ which defines one differential operator $\mathcal{L}_1 = \mathcal{L}(l^1)$ annihilating the periods on the mirror. For this operator we do not get an exact term, and we find

$$\left(\prod_{i=1}^{5}\vartheta_i + z\prod_{i=1}^{5}(\vartheta_0 - i)\right)\ \Omega = 0\ . \tag{3.7}$$

[11] This is what one would expect intuitively from the formulation of mirror symmetry as T duality [37].

Furthermore the operators $\mathcal{Z}_k$ give rise to the relations

$$\sum_{i=0}^{5} \vartheta_i \; \Omega = 0, \qquad (\vartheta_i - \vartheta_5) \; \Omega = d\omega_i \ , \quad i = 1, ..., 4, \tag{3.8}$$

with

$$\omega_i = (-)^{i+1} \mathrm{Res}_P \; \frac{a_0}{P} \prod_{\substack{j=1 \\ j\neq i}}^{4} \frac{dX_j}{X_j} \ . \tag{3.9}$$

Equation (3.8) expresses the torus invariance of the period integrals in absolute cohomology and implies that the integrals depend only on the single invariant complex modulus z_1 defined as in eq. (3.4). In relative cohomology, the exact terms on the r.h.s. descend to non-trivial 2-forms on D by the equivalence relation

$$H^3(Z^*, D) \ \ni \ (\Xi, \xi) \sim (\Xi + d\alpha, \xi + i^*\alpha - d\beta) \ , \tag{3.10}$$

where $i : D \hookrightarrow Z^*$ is the embedding. The exact pieces in (3.8) may give rise to boundary terms that break the torus symmetry and introduce an additional dependence on moduli $\hat{z}_\alpha$ associated with the geometry of the embedding of D.

To proceed we need to specify the family of hypersurfaces D. As in refs. [7,9] we consider a simple 1-parameter family D_1 of hypersurfaces defined by a linear equation, which can be put into the standard form

$$D_1 : \ Q = 1 + X_1 = 0 \ , \tag{3.11}$$

by a coordinate transformation on X_1.[12] In order to determine the preserved torus symmetries we examine the boundary contributions (3.8) with respect to the hypersurfaces D_1 by evaluating the pullbacks of the two forms (3.9). One finds $i^*\omega_k = 0$ for $k = 2, 3, 4$ and

$$i^*\omega_1 = \mathrm{Res}_{P_D} \frac{a_0}{P_D} \prod_{i=2}^{4} \frac{dX_i}{X_i} \ = \mathrm{Res}_{P,Q} \ \frac{a_0 X_1}{P\,Q} \prod_{k=1}^{4} \frac{dX_k}{X_k} \ , \tag{3.12}$$

where $P_D = P(Z^*)|_{Q=0} = (a_0 - a_1) + a_2 X_2 + a_3 X_3 + a_4 X_4 - a_5 (X_2 X_3 X_4)^{-1}$. In the second equation above, we represented the pull-back as a double residue in $P = 0$ and $Q = 0$ in

[12] At first glance it seems that we have chosen a rigid hypersurface D_1. However, as we vary the embedding of the Calabi-Yau manifold Z^* in the ambient toric space, we effectively change the hypersurface D_1 in in the Calabi-Yau manifold Z^*.

the ambient space as in ref. [7], which allows for a direct evaluation of periods as in ref. [26] and is convenient also for higher degree hypersurfaces.

In the presence of the hypersurface D_1 the torus action $X_1 \to \lambda X_1$ with $\lambda \in \mathbf{C}^*$ generated by the operator $\mathcal{Z}_1$ is broken, whereas the remaining torus symmetries, associated to the operators $\mathcal{Z}_0, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4$, are preserved.[13] From these four differential operators for the six parameters a_i it follows that the family $D_1 \subset Z^*$ depends precisely on the two moduli

$$z = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}, \qquad \hat{z} = -\frac{a_1}{a_0} \ , \tag{3.13}$$

together with the logarithmic derivatives

$$\theta := z\,\partial_z = \vartheta_5 \ , \qquad \hat{\theta} := \hat{z}\,\partial_{\hat{z}} = \mathcal{Z}_1 = \vartheta_1 - \vartheta_5 \ . \tag{3.14}$$

Here z is the complex structure modulus of Z^* and $\hat{z}$ is the open-string position modulus parametrizing the position of the brane. This analysis justifies in retrospect that the defined family of hypersurfaces (3.11) indeed depends on a single open-string modulus.

The next task is to determine the differential operators $\mathcal{L}$ of the extended GKZ hypergeometric system associated to the periods of the relative (3,0) form representative

$$\underline{\Omega} := (\Omega, 0) = \left(\mathrm{Res}_P \frac{a_0}{P} \prod_k \frac{dX_k}{X_k} \ ,0\right) \ . \tag{3.15}$$

Due to eq. (3.7) the operator $\mathcal{L}_1$ annihilates the three form Ω even on the level of forms (and not just on the level of the absolute three-form cohomology), and therefore the operator $\mathcal{L}_1$ annihilates also the relative three form $\underline{\Omega}$. In the extended relative cohomology setting, however, there is an additional differential operator $\mathcal{L}_2$ governing the functional dependence of the exact piece $d\omega_1$. It is straightforward to check that the (2,0) form $i^*\omega_1$ of the hypersurface D_1 defined in eq. (3.12) obeys

$$(\vartheta_1 - \hat{z}(\vartheta_0 - 1))\, i^*\omega_1 = 0 \ .$$

Together with the relation

$$\theta_2 \underline{\Omega} = \mathcal{Z}_1(\Omega, 0) = (d\omega_1, 0) \sim (0, -i^*\omega_1) \ , \tag{3.16}$$

[13] Alternatively one may express the independence of the periods on extra parameters in (3.11) by a modified differential operator $\tilde{Z}_1$, see also the discussion below eq. (3.19).

associated to the broken torus action, we determine the second operator $\mathcal{L}_2$ to be

$$\mathcal{L}_2 = (\vartheta_1 - \hat{z}(\vartheta_0 - 1))(\vartheta_1 - \vartheta_5) \ .$$

Thus in summary we find that the relative (3,0)-form cohomology class, represented by the relative three-form $\underline{\Omega}$, is annihilated by the six differential operators

$$\begin{aligned} &\mathcal{L}_a\underline{\Omega} \sim 0 \ , \quad a = 1,2 \ , \\ &\mathcal{Z}_k\underline{\Omega} \sim 0 \ , \quad k = 0,2,3,4 \ . \end{aligned} \tag{3.17}$$

With the help of the differential operators $\mathcal{Z}_k, k = 0,2,3,4$, it is straightforward to derive the two extended GKZ operators $\mathcal{L}_a, a = 1,2$, in terms of the moduli z and $\hat{z}$ and their logarithmic derivatives θ and $\hat{\theta}$

$$\begin{aligned} \mathcal{L}_1 &= (\theta + \hat{\theta})\theta^4 + z\prod_{i=1}^{5}(-5\theta - \hat{\theta} - i) =: \mathcal{L}_1^{bulk} + \mathcal{L}_1^{bdry}\hat{\theta} \quad , \\ \mathcal{L}_2 &= \big((\theta + \hat{\theta}) - \hat{z}(-5\theta - \hat{\theta} - 1)\big)\hat{\theta} =: \mathcal{L}_2^{bdry}\hat{\theta} \ . \end{aligned} \tag{3.18}$$

Here $\mathcal{L}_1^{bulk} = \theta^5 + z\prod_{i=1}^5(-5\theta - i)$ is the $\hat{\theta}$-independent GKZ operator of the quintic and the operators $\mathcal{L}_a^{bdry}$ are always accompanied by at least one derivative $\hat{\theta}$ and thus are only sensitive to boundary contributions

$$0 \ = \ \mathcal{L}_a \int_{\gamma_\Sigma} \underline{\Omega} \ = \ \mathcal{L}_a^{bulk} \int_{int(\gamma_\Sigma)} \Omega - \mathcal{L}_a^{bdry} \int_{\partial\gamma_\Sigma} i^*\omega_1 \ , \quad a = 1,2 \ , \tag{3.19}$$

with $\mathcal{L}_2^{bulk} \equiv 0$.

The significance of the interplay of the operator $\hat{\theta}$ with the operator $\mathcal{L}_a^{bdry}$ is reflected in eq. (3.16). We observe that the (linear combinations) of boundary operators $\mathcal{L}_a^{bdry}$, which are not accompanied with a non-vanishing bulk operator $\mathcal{L}_a^{bulk}$, become the GKZ operators of the periods localizing on the hypersurface D_1. As noted in ref. [9], the Hodge variation on the hypersurface $P_D = 0$ is isomorphic to that of the mirror of the quartic K3 surface and $i^*\omega_1$ is a representative for the holomorphic (2,0) form. The associated K3 periods $\int i^*\omega_1$ are precisely annihilated by these GKZ operators arising form the boundary sector.

Thus we have obtained two differential operators $\mathcal{L}_a, \ a = 1,2$, in eq. (3.18) annihilating the relative periods. These operators can be rewritten in a concise form by realizing that they represent the differential operators $\mathcal{L}(l)$ for a different GKZ system of the type (3.3) specified by the two linear relations

$$\tilde{l}^1 = (-5;1,1,1,1,1,0,0), \qquad \tilde{l}^2 = (-1;1,0,0,0,0,1,-1) \ , \tag{3.20}$$

together with the five-dimensional enhanced toric polyhedron $\tilde{\Delta}$ with the integral vertices [9]

$$\begin{array}{llll}\tilde{\nu}_0 = (0,0,0,0,0)\ , & \tilde{\nu}_1 = (1,0,0,0,0)\ , & \tilde{\nu}_2 = (0,1,0,0,0)\ , & \tilde{\nu}_3 = (0,0,1,0,0)\ , \\ \tilde{\nu}_4 = (0,0,0,1,0)\ , & \tilde{\nu}_5 = (-1,-1,-1,-1,0)\ , & \tilde{\nu}_6 = (0,0,0,0,1)\ , & \tilde{\nu}_7 = (1,0,0,0,1)\ .\end{array} \tag{3.21}$$

This polyhedron defines a CY four-fold X^*, which is dual to the brane compactification in the sense of ref. [8]. A systematic construction, which associates a compact dual F-theory four-fold to a mirror pair of toric branes defined as in ref. [3], is relegated to sect. 7.

Note that the enhanced polyhedron $\tilde{\Delta}$ gives rise to six operators $\tilde{\mathcal{Z}}_k, k = 0, \ldots, 5$, with $\tilde{\mathcal{Z}}_k = \mathcal{Z}_k$ for $k = 0, 2, 3, 4$. The two additional operators $\tilde{\mathcal{Z}}_1$ and $\tilde{\mathcal{Z}}_5$ guarantee that the functional dependence on the local coordinates $\tilde{z}_a(\tilde{l}^a)$ defined by the general relation (3.4) also coincide with the moduli of the relative cohomology problem defined in eq. (3.13). Moreover, the GKZ operators $\tilde{\mathcal{L}}_a = \mathcal{L}(\tilde{l}^a)$ obtained from the general expression

$$\mathcal{L}(l) = \prod_{k=1}^{l_0}(\vartheta_0 - k)\prod_{l_i>0}\prod_{k=0}^{l_i-1}(\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0}(\vartheta_0 - k)\prod_{l_i<0}\prod_{k=0}^{-l_i-1}(\vartheta_i - k) \tag{3.22}$$

coincide with the two operators $\mathcal{L}_a$ of the relative cohomology problem in the local coordinates defined by (3.4).

$$\mathcal{L}_a(z,\hat{z}) = \tilde{\mathcal{L}}_a(\tilde{z}_1 = z, \tilde{z}_2 = \hat{z})\ .$$

In fact one can show that all differential operators $\mathcal{L}(l)$ for l a linear combination of $\tilde{l}^1, \tilde{l}^2$ also annihilate the relative periods.[14] The coincidence of the system of differential operators for the periods on the relative cohomology group $H^3(Z^*, D)$ and the GKZ system for the dual four-folds constructed by the method of ref. [9] holds more generally for relative cohomology groups associated with the class of toric branes on toric hypersurfaces defined in ref. [3].

4. Gauss-Manin connection and integrability conditions

The Picard-Fuchs system for the relative periods derived in the previous section captures the variation of Hodge structure on the relative cohomology group $H^3(Z^*, D)$. In this section we work out some relations and predictions for mirror symmetry for a family of A- and B-branes from the associated Gauss-Manin connection.

[14] For the quintic the additional operators are of the form $\mathcal{L}^{bdry}\theta_2$, where $\mathcal{L}^{bdry} i^*\omega_1 = 0$ modulo exact 2-forms on D_1.

4.1. Gauss-Manin connection on the open-closed deformation space $\mathcal{M}$

Geometrically we can view $H^3(Z^*, D)$ as the fiber of a complex vector bundle over the open-closed deformation space $\mathcal{M}$. As the relative cohomology group[15] H^3 depends only on the topological data, the fiber is up to monodromy constant over $\mathcal{M}$, and there is a trivially flat connection, ∇, called the Gauss-Manin connection. The Hodge decomposition $H^3 = \oplus_{p=0}^3 H^{3-p,p}$ varies over $\mathcal{M}$, as the definition of the Hodge degree depends on the complex structure. The Hodge filtrations F^p

$$H^3 = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0 \ , \qquad F^p = \oplus_{q \geq p} H^{q,3-q} \subset H^3 \ ,$$

define holomorphic subbundles $\mathcal{F}^p$ whose fibers are the subspaces $F^p \subset H^3$. The action of the Gauss-Manin connection ∇ on these subbundles has the property $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes T^*_{\mathcal{M}}$, known as Griffiths transversality.

Concretely, the mixed Hodge structure on the relative cohomology space $H^3(Z^*, D)$ looks as follows. The Hodge filtrations are

$$\begin{aligned}
F^3 =&\ H^{3,0}(Z^*, D) &&= H^{3,0}(Z^*) \ , \\
F^2 =&\ F^3 \oplus H^{2,1}(Z^*, D) &&= F^3 \oplus H^{2,1}(Z^*) \oplus H^{2,0}_{var}(D) \ , \\
F^1 =&\ F^2 \oplus H^{1,2}(Z^*, D) &&= F^2 \oplus H^{1,2}(Z^*) \oplus H^{1,1}_{var}(D) \ , \\
F^0 =&\ F^1 \oplus H^{0,3}(Z^*, D) &&= F^1 \oplus H^{0,3}(Z^*) \oplus H^{0,2}_{var}(D) \ ,
\end{aligned} \tag{4.1}$$

where the equations to the right display the split $H^3(Z^*, D) \simeq \ker\big(H^3(Z^*) \to H^3(D)\big) \oplus \operatorname{coker}\big(H^2(Z^*) \to H^2(D)\big)$. The weight filtration is defined as

$$W_2 = 0 \ , \qquad W_3 = H^3(Z^*) \ , \qquad W_4 = H^3(Z^*, D) \ ,$$

such that the quotient spaces $W_3/W_2 \simeq H^3(Z^*)$ and $W_4/W_3 \simeq H^2(D)$ define pure Hodge structures. Variations in the closed (δ_z) and open ($\delta_{\hat{z}}$) string sector act schematically as

$$\begin{tikzcd}
F^3 \cap W_3 \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat{z}}"] & F^2 \cap W_3 \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat{z}}"] & F^1 \cap W_3 \arrow[r, "\delta_z"] \arrow[dr, "\delta_{\hat{z}}"] & F^0 \cap W_3 \\
 & F^2 \cap (W_4/W_3) \arrow[r, "{\delta_z, \delta_{\hat{z}}}"] & F^1 \cap (W_4/W_3) \arrow[r, "{\delta_z, \delta_{\hat{z}}}"] & F^0 \cap (W_4/W_3)
\end{tikzcd} \tag{4.2}$$

The variation of the Hodge structure over $\mathcal{M}$ can be measured by the period matrix

$$\underline{\Pi}^{\Sigma}_{A} = \int_{\gamma_\Sigma} \alpha_A \ , \qquad \alpha_A \in H^3 \ ,$$

[15] Letters without arguments refer to relative cohomology over $\mathbf{C}$, e.g. $H^3 = H^3(Z^*, D; \mathbf{C})$.

where γ_Σ is a fixed topological basis for $H_3(Z^*,D)$ and $\{\alpha_A\}$ with $A=1,...,\dim(H^3)$ denotes a basis of relative 3-forms. One may choose an ordered basis $\{\alpha_A^{(q)}\}$ adapted to the Hodge filtration, such that the subsets $\{\alpha_A^{(q')}\}$, $q'\le q$ span the spaces F^{3-q} for $q=0,...,3$.

To make contact between the Hodge variation and the B-model defined at a point $m\in\mathcal{M}$, the Gauss-Manin connection has to be put into a form compatible with the chiral ring properties of a SCFT. Chiral operators of definite $U(1)$ charge are identified with forms of definite Hodge degree, which requires a projection onto the quotient spaces F^p/F^{p+1} at the point m. Moreover, the canonical CFT coordinates t_a, centered at $m\in\mathcal{M}$, should flatten the connection ∇ and we require

$$\nabla_a\alpha_A^{(q)}(m)=\partial_{t_a}\alpha_A^{(q)}(m)\overset{!}{=}(C_a(t)\cdot\alpha_A^{(q)})(m)\ \in F^{3-q-1}/F^{3-q}|_m\ . \tag{4.3}$$

The second equation is an important input as it expresses the non-trivial fact, that in the CFT, an infinitesimal deformation in the direction t_a is generated by an insertion of (the descendant) of a chiral operator $\phi_a^{(1)}$ in the path integral, which in turn can be described by a naive multiplication by the operator $\phi_a^{(1)}$ represented by the connection matrix $C_a(t)$. The above condition assumes that such a simple relation holds on the full open-closed deformation space for all deformations in F^2/F^3. Thus $\phi_a^{(1)}$ can be either a bulk field of left-right $U(1)$ charge $(1,1)$ or a boundary operator of total $U(1)$ charge 1. The consistency of the results obtained below with this ansatz and the correct matching with the CFT deformation space discussed in sect. 6 provides evidence in favor of a proper CFT realization of this structure.

Phrased differently, we consider the $\alpha_A^{(q)}$ as flat sections of an "improved" flat connection D_a in the sense of refs. [38,39]

$$D_a\alpha_A^{(q)}=0,\qquad D_a=\partial_{z_a}-\Gamma_a(z)-C_a(z),\qquad [D_a,D_b]\ =\ 0\ ,$$

where z_a are local coordinates on $\mathcal{M}$, the connection terms $\Gamma_a(z)$ and $C_a(z)$ are maps from $\mathcal{F}^{3-q}$ to $\mathcal{F}^{3-q}$ and $\mathcal{F}^{3-q-1}$, respectively, and $\Gamma_a(z)$ vanishes in the canonical CFT coordinates t_a.[16]

Instead of working in generality, we study the Gauss-Manin connection for the relative cohomology group on the two parameter family of branes on the quintic defined by

[16] See sect. 2.6 of ref. [39] for the definition of canonical coordinates from the (closed-string) CFT point of view.

(3.11). We consider a large volume phase with moduli (3.4) defined by the following linear combination of charge vectors (3.20):

$$\tilde{l}^1 = (-4; 0, 1, 1, 1, 1, -1, 1), \qquad \tilde{l}^2 = (-1; 1, 0, 0, 0, 0, 1, -1) \ ,$$

A complete set of differential operators derived from eq. (3.22) is given by

$$\begin{aligned}
\mathcal{L}_1 &= \theta_1^4 + (4z_1(\theta_1 - \theta_2) - 5z_1 z_2(4\theta_1 + \theta_2 + 4)) \prod_{i=1}^{3}(4\theta_1 + \theta_2 + i) \ , \\
\mathcal{L}_2 &= \theta_2(\theta_1 - \theta_2) + z_2(\theta_1 - \theta_2)(4\theta_1 + \theta_2 + 1) \ , \\
\mathcal{L}_3 &= \theta_1^3(\theta_1 - \theta_2) + (4z_1 \prod_{i=1}^{3}(4\theta_1 + \theta_2 + i) + z_2\theta_1^3)(\theta_1 - \theta_2) \ .
\end{aligned} \tag{4.4}$$

Computing the ideal generated by the $\mathcal{L}_k$ acting on $\underline{\Omega}$ shows that H^3 is a seven-dimensional space spanned by the multiderivatives $(1, \theta_1, \theta_2, \theta_1^2, \theta_1\theta_2, \theta_1^3, \theta_1^2\theta_2)$ of $\underline{\Omega}$. The dimensions $d_q = \dim(F^{3-q}/F^{3-q+1})$ are $1, 2, 2, 2$ for $q = 0, 1, 2, 3$, respectively. The $d_1 = 2$ directions tangent to $\mathcal{M}$ represent the single complex structure deformation $z = z_1 z_2$ of the mirror quintic Z^* and the parameter $\hat{z} = z_2$ parametrizing the family of hypersurfaces.

To implement a CFT like structure at a point $m \in \mathcal{M}$, one may take linear combinations of the multiderivatives acting on $\underline{\Omega}$ to obtain ordered bases $\{\alpha_A^{(q)}\}$ and $\{\gamma_\Sigma\}$ which bring the period matrix into a block upper triangular form[17]

$$\underline{\Pi}_\Sigma^A = \begin{pmatrix} 1 & * & * & * \\ 0 & 1\!\!1_{d_1 \times d_1} & * & * \\ 0 & 0 & 1\!\!1_{d_2 \times d_2} & * \\ 0 & 0 & 0 & 1\!\!1_{d_3 \times d_3} \end{pmatrix} \ . \tag{4.5}$$

Griffiths transversality then implies that in the local coordinates at a point of maximal unipotent monodromy

$$\begin{pmatrix} \nabla\alpha_1^{(0)} \\ \nabla\alpha_2^{(1)} \\ \nabla\alpha_3^{(1)} \\ \nabla\alpha_4^{(2)} \\ \nabla\alpha_5^{(2)} \\ \nabla\alpha_6^{(3)} \\ \nabla\alpha_7^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(1)} & 0 & 0 \\ 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(2)} & 0 \\ 0 & 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(3)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1^{(0)} \\ \alpha_2^{(1)} \\ \alpha_3^{(1)} \\ \alpha_4^{(2)} \\ \alpha_5^{(2)} \\ \alpha_6^{(3)} \\ \alpha_7^{(3)} \end{pmatrix} , \tag{4.6}$$

[17] It is understood, that all entries in the following matrices are block matrices operating on the respective subspaces of definite $U(1)$ charge, with dimensions determined by the numbers d_q.

where the moduli-dependent matrices $M_a^{(q)}$ of dimension $d_{q-1} \times d_q$ are derivatives of the entries of $\underline{\Pi}$ in eq.(4.5). The above expression is written in logarithmic variables $\log(z_a)$, anticipating the logarithmic behavior of the periods at the point of maximal unipotent monodromy centered at $z_a = 0$. In local coordinates x_a centered at a generic point $m \in \mathcal{M}$, the periods are analytic in x_a and dz_a/z_a should be replaced with dx_a.

The left upper block can be brought into the form

$$\sum_{a=1}^{2} \frac{dz_a}{z_a} M_a^{(1)} = \left(\frac{dq_1}{2\pi i\, q_1}, \frac{dq_2}{2\pi i\, q_2}\right)$$

by the variable transformation

$$q_a(z) = \exp(2\pi i\, \underline{\Pi}_1^{a+1}(z)) \ . \tag{4.7}$$

It has been proposed in ref. [6], that eq.(4.7) represents the mirror map between the A-model Kähler coordinates $t_a = \frac{1}{2\pi i} \ln(q_a)$ on the open-closed deformation space of an A-type compactification (Z, L) and the coordinates z_a on the complex structure moduli space of an B-type compactification (Z^*, E) near a large complex structure point. We propose that the above flatness conditions defines more generally the mirror map between the open-closed deformation spaces for any point $m \in \mathcal{M}$.[18] It is worth stressing that the mirror map defined by the above flatness argument coincides with the mirror map obtained earlier in refs. [3,41] for non-compact examples by a physical argument, using domain wall tensions and the Ooguri-Vafa expansion at a large complex structure point. This coincidence can be viewed as experimental evidence for the existence of a more fundamental explanation of the observed flat structure from the underlying topological string theory, as advocated for in this note.

Identifying $\alpha_1^{(0)}$ with the unique operator $\phi^{(0)} = 1$ and the $\alpha_{a+1}^{(1)}$ with the charge one operators $\phi_a^{(1)}$ associated with the flows parametrized by $\log(q_a)$, eq. (4.6) implements the CFT relation

$$\nabla_{q_a} \phi^{(0)} = \phi_a^{(1)} = \phi_a^{(1)} \cdot \phi^{(0)} \ ,$$

discussed below (4.3). The above series of arguments and manipulations is standard material in closed string mirror symmetry and led to the deep connection between the geometric Hodge variations of CY three-folds in the B-model and A-model quantum cohomology

[18] A non-trivial example will be described in ref. [40].

on the mirror.[19] After the variable transformation (4.7) and restricting to the subspace $H^3(Z^*)$ describing the complex moduli space $\mathcal{M}_{CS}$ of the mirror quintic, eq.(4.6) becomes

$$\begin{pmatrix} \nabla\alpha_{cl}^{(0)} \\ \nabla\alpha_{cl}^{(1)} \\ \nabla\alpha_{cl}^{(2)} \\ \nabla\alpha_{cl}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{2\pi i\, q} & 0 & 0 \\ 0 & 0 & C(q)\frac{dq}{2\pi i\, q} & 0 \\ 0 & 0 & 0 & -\frac{dq}{2\pi i\, q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{cl}^{(0)} \\ \alpha_{cl}^{(1)} \\ \alpha_{cl}^{(2)} \\ \alpha_{cl}^{(3)} \end{pmatrix} \,, \tag{4.8}$$

with $\alpha_{cl}^{(p)} \in H^{3-p,p}(Z^*,\mathbf{C})$ and $q = q_1 q_2 = e^{2\pi i t}$. Under mirror symmetry these data get mapped to the Kähler volume t and the so-called Yukawa coupling $C(q) = 5 + \mathcal{O}(q)$, which describes the classical intersection and the Gromov-Witten invariants on the quintic. In the CFT, the quantities $C(q)$ represent the moduli-dependent structure constants of the ring of chiral primaries defined in ref. [42].

The point which we are stressing here is that at least part of these concepts continue to make sense for the Hodge variation (4.6) on the full relative cohomology space $H^3(Z^*,D)$ over the open-closed deformation space $\mathcal{M}$ fibered over $\mathcal{M}_{CS}$. More importantly, the Hodge theoretic definition of mirror symmetry described above gives correct results for the open string analogues of the Gromov-Witten invariants in those cases, where results have been obtained by different methods, such as space-time arguments involving domain walls [3,41].[20]

In this sense, the existence of a flat structure observed above, and the agreement of the Hodge theoretic results with other methods, if available, urges for a proper CFT description of the deformation families defined over $\mathcal{M}$ and an appropriate open-string extension of A-model quantum cohomology. In the following we collect further evidence in favor of an interesting integrable structure on the open-closed deformation space, working in the B-model.

4.2. Integrability conditions

The correlation functions of the topological family of closed-string CFTs satisfy the famous WDVV integrability condition [44,45]. In the context of the B-model on a CY three-fold, this condition becomes part of $\mathcal{N}=2$ special Kähler geometry of the complex

[19] See refs. [12,29] for background material and a comprehensive list of references.

[20] See [43,6,16,7,9,26] for various examples.

structure moduli space, which implies, amongst others, the existence of a single holomorphic prepotential $\mathcal{F}$ that determines all entries of the period matrix in the canonical CFT coordinates t_a.

There exists no prepotential for the period matrix (4.5) on the relative cohomology group $H^3(Z^*, D)$, but certain aspects of the $\mathcal{N}=2$ special geometry of the closed-string sector $H^3(Z^*) \subset H^3(Z^*, D)$ generalize to the larger cohomology space, justifying the term $\mathcal{N}=1$ special geometry [8,6].[21] Some aspects of this $\mathcal{N}=1$ special geometry have been worked out for non-compact Z^* in [6,16] and we add here some missing pieces for the compact case. In the following we work at a "large complex structure point" $m_0 \in \mathcal{M}$ of maximal unipotent monodromy. The existence of such points m_0 follows from the general property of the GKZ systems described in sect. 3.

We start from the following general ansatz for the 7-dimensional period vector of the holomorphic 3-form

$$\underline{\Pi}_1^\Sigma = \big(1,\ t,\ \hat{t},\ F_t(t),\ W(t,\hat{t}),\ F_0(t),\ T(t,\hat{t})\ \big)\ , \tag{4.9}$$

where t is the closed- and $\hat{t}$ the open-string deformation, related to the flat normal crossing divisor coordinates $(t_1(z_1,z_2), t_2(z_1,z_2))$ of eq. (4.7) by the linear transformation $t = t_1 + t_2$, $\hat{t} = t_2$. The subset of periods in the closed-string sector is determined by the prepotential $\mathcal{F}$ as $(1, t, F_t = \partial_t \mathcal{F}, F_0(t) = 2\mathcal{F}(t) - t\partial_t \mathcal{F})$ and depends only on t. The additional periods $(\hat{t}, W(t,\hat{t}), T(t,\hat{t}))$ are so far arbitrary functions, except that the leading behavior at m_0 at $z_a = 0$ is, schematically,

$$t, \hat{t} \sim \log(z_.), \qquad F_t, W \sim \log^2(z_.), \qquad F_0, T \sim \log^3(z_.)\ .$$

The function W is in some sense the closest analogue of the closed string prepotential and indeed has been conjectured to be a generating function for the open-string disc invariants in [6,7].

For an appropriate choice of basis $\{\alpha_A^{(q)}\}$, the period matrix takes the upper triangular form (4.5) with entries

$$(\underline{\Pi}) = \begin{pmatrix} 1 & t & \hat{t} & F_t & W & F_0 & T \\ 0 & 1 & 0 & F_{t,t} & W_{,t} & F_{0,t} & T_{,t} \\ 0 & 0 & 1 & 0 & W_{,\hat{t}} & 0 & T_{,\hat{t}} \\ 0 & 0 & 0 & 1 & 0 & -t & \mu \\ 0 & 0 & 0 & 0 & 1 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{4.10}$$

[21] $\mathcal{N}=1,2$ denotes the number of 4d space-time supersymmetries of the CY compactification of the physical type II string to four dimensions with and without branes.

where the derivatives w.r.t. t and $\hat{t}$ are denoted by subscripts and the functions μ and ρ are defined by

$$\mu = \frac{W_{,t\hat{t}}T_{,tt} - W_{,tt}T_{,t\hat{t}}}{CW_{,t\hat{t}}}, \qquad \rho = \frac{T_{,t\hat{t}}}{W_{,t\hat{t}}}, \qquad C = \mathcal{F}_{,ttt} = F_{t,tt} \ .$$

The connection matrices read

$$M_t \ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & W_{,tt} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \mu_t \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ M_{\hat{t}} \ = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,\hat{t}\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.11}$$

The integrability condition $\partial_t M_{\hat{t}} - \partial_{\hat{t}} M_t + [M_{\hat{t}}, M_t] = 0$ implies that

$$\rho(t,\hat{t}) = aW_{,\hat{t}} + b \ , \qquad \mu(t,\hat{t}) = aC^{-1}(t) \left(\int (W_{,t\hat{t}}^2 - W_{,tt}W_{,\hat{t}\hat{t}})\, d\hat{t} + g(t) \right) \ , \tag{4.12}$$

with a, b some complex constants and $g(t)$ an undetermined function. The relation (4.3) then implies that the period T is of the form

$$T(t,\hat{t}) = \int \left(\frac{a}{2} W_{,\hat{t}}^2 + bW_{,\hat{t}} \right) d\hat{t} + f(t) \ , \tag{4.13}$$

with $\partial_t^2 f(t) = g(t)$. The integrability condition (4.13) determines the top period in the open-string sector in terms of the other periods, up to the function $f(t)$. In this sense it is similar to the relation in the closed-string sector, that determines the top period F_0 in terms of the other periods. The integration constants can be fixed by determining the leading behavior of the periods in the large volume limit, as we will do in app. A for the quintic example.

The above argument and the integrability relation (4.13) applies to any two parameter family with one closed- and one open-string modulus and can be straightforwardly generalized to more parameter cases. For a given geometry, such as the quintic family described by the operators $\mathcal{L}_k$ in (4.4), one can of course reach the same conclusion by studying the explicit solutions and also determine the function $f(t)$. As noticed below eq. (3.19), the relative forms in the open-string sector of this family can be associated with the Hodge variation on a quartic K3 surface. The period vector $\vec{\pi}$ of the K3 surface is spanned by the solutions $\partial_{\hat{t}}(\hat{t}, W, T) = (1, W_{,\hat{t}}, T_{,\hat{t}})$ and the integrability condition (4.13) represents an

algebraic relation $\vec{\pi}^T \hat{\eta} \vec{\pi} = 0$ amongst the K3 periods, where $\hat{\eta}$ is the intersection matrix. We come back to the intersection form $\hat{\eta}$ when we discuss the topological metric in sect. 7.

The above discussion was essentially independent of the choice of the large complex structure point m_0 and a similar argument for other m shows that the integrability condition (4.13) holds for any $m \in \mathcal{M}$ in the local coordinates defined by (4.7).

A cautious remark

The similarities of the above arguments with those in the case of closed string mirror symmetry may have obscured the fact that one crucial datum is still incomplete: the topological metric η on the open-closed state space. In the closed-string sector, the topological metric η_{cl} is given by the classical intersection matrix on $H_3(Z^*)$ and its knowledge permits, amongst others, the determination of the true geometric periods as a particular linear combinations of the solutions to the Picard-Fuchs equations. More importantly, the topological metric is needed to complete the argument that identifies $C(q)$ in eq.(4.8) with the structure constants of the chiral ring of ref. [42], as well as to access the non-holomorphic and higher genus sector of the theory using the tt^* equations [38] and the holomorphic anomaly equation [39]. Our present lack of a precise understanding of the topological metric in the open-string sector renders the following sections somewhat fragmentary.

In the next section we derive predictions for disc invariants by fixing the geometric periods in a way that avoids the knowledge of the topological metric. Some observations on the other issues mentioned above, including a proposal for the full topological metric, will be discussed in sects. 6 and 7.

5. Large radius invariants for the A-model

The geometry of A-branes is notorically more difficult to study than that of the B-branes. For the type of B-branes studied above, the A-model mirror geometry can be in principle obtained from the toric framework of ref. [3]. The conjectural family of A-branes mirror to the B-brane family studied above, is defined on the quintic hypersurface Z in the toric ambient space $W = \mathcal{O}(-5)_{\mathbf{P}^4}$ with homogeneous coordinates

$$\begin{array}{cccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ -5 & 1 & 1 & 1 & 1 & 1 \end{array}$$

The Kähler moduli (t_1, t_2) mirror to the complex structure coordinates (z_1, z_2) are defined by the equations

$$\sum_{i=1}^{5} |x_i|^2 - 5|x_0|^2 = \mathrm{Im}\, t = \mathrm{Im}\, t_1 + \mathrm{Im}\, t_2, \qquad |x_1|^2 - |x_0|^2 = \mathrm{Im}\, t_2 \ , \tag{5.1}$$

with $\mathrm{Im}\, t_1, \mathrm{Im}\, t_2 > 0$. The first constraint holds on all of Z and the Kähler modulus t describes the closed string deformation, the overall Kähler volume of Z. The second constraints holds only on the Lagrangian submanifold L and describes the open-string deformation.

The toric framework of ref. [3] gives an explicit description of the geometry of Lagrangian subspaces in the ambient space W, which has been used to study an interesting class of non-compact branes for CY ambient spaces, see e.g. [3,41,6]. The clear geometric picture of the toric description is lost for hypersurfaces and the searched for subspace $L \subset Z$ carrying the mirror A-brane has no simple description, at least at general points in the (full) moduli space and to our knowledge. However, by the homological mirror symmetry conjecture [1], we expect that a corresponding A-brane, which is mirror to the B-brane given in terms of the discussed divisor together with its curvature 2-form, should be present in the A-model geometry. Clearly, in order to complete our picture a constructive recipe of mapping B-branes to the corresponding A-branes for compact CY geometries is desirable. We hope to come back to this issue elsewhere.

Since the A-model geometry is naively independent of the complex structure moduli, one is tempted to choose a very special form of the hypersurface constraint to simplify the geometry. In ref. [46] it is shown how the number 2875 of lines in the generic quintic can be determined from the number of lines in a highly degenerate quintic, defined by the hypersurface constraint

$$P(Z_\alpha) = p_1 . p_4 + \alpha\, p_5, \qquad \alpha \in \Delta \ . \tag{5.2}$$

Here α is a parameter on the complex disc Δ and the p_k are degree k polynomials in the homogeneous coordinates of $\mathbf{P}^4$. At the point $\alpha = 0$ the quintic splits into two components of degree one and four. Katz shows, that there are 1600+1275=2875 holomorphic maps to lines in the two components of the central fiber that deform to the fiber at $\alpha \neq 0$.

The $N_1 = 2875$ curves of degree one contribute to the tension $\mathcal{T}$ of a D4-brane wrapping the 4-cycle $\Gamma = H \cap Z$ as

$$\mathcal{T} = -\frac{5}{2} t^2 + \frac{1}{4\pi^2} \left(2875 \sum_k \frac{q^k}{k^2} + 2 \cdot 609250 \sum_k \frac{q^{2k}}{k^2} + ... \right)$$

where $q = \exp(2\pi i t)$, H is the hyperplane class and the dots denote linear and constant terms in t as well as instanton corrections from maps of degree $d > 2$. In the singular CY the generic 4-cycle splits into two components and one expects two separate contributions

$$\mathcal{T}^{(1)} = c^{(1)}\, t^2 + \frac{N_d^{(1)}}{4\pi^2} \sum_k \frac{q^{dk}}{k^2} + ..., \qquad \mathcal{T}^{(2)} = \mathcal{T} - \mathcal{T}_1 \ ,$$

with $N_1^{(1)} = 1600$ and $N_1^{(2)} = 1275$.

As explained in ref. [46], there are other genus zero maps to the two components, that develop nodes at the intersection locus $p_1 = p_4 = 0$ upon deformation, and they do not continue to exist as maps from S^2 to S^2. The idea is that in the presence of the Lagrangian A-brane on the degenerate quintic, the nodes of the spheres can open up to become the boundary of holomorphic disc instantons ending on L. Indeed the two independent double logarithmic solutions of the Picard-Fuchs system (4.4) can be written in the flat coordinates (4.7) as

$$\begin{aligned}
\mathcal{T}^{(1)} &= -2t^2 + \frac{1}{4\pi^2} \sum_k \ \frac{1}{k^2}\big(1600q^k + 2\cdot 339800q^{2k} + ...\big) + \mathcal{T}^{(o)}(t_1,t_2) \\
\mathcal{T}^{(2)} &= -\frac{1}{2}t^2 + \frac{1}{4\pi^2} \sum_k \ \frac{1}{k^2}\big(1275q^k + 2\cdot 269450q^{2k} + ...\big) - \mathcal{T}^{(o)}(t_1,t_2)
\end{aligned} \tag{5.3}$$

showing the expected behavior and adding up to the closed-string period $\mathcal{T}$. The split of the degree two curves, $N_2 = 609250 = 339800 + 269450 = (258200 + \frac{1}{2}163200) + (187850 + \frac{1}{2}163200)$ is compatible with the results of ref. [47].

The extra contribution $\mathcal{T}^{(o)}(t_1,t_2)$ can be written as

$$\mathcal{T}^{(o)}(t_1,t_2) = 4tt_2 - 2t_2^2 + \frac{1}{4\pi^2} \sum_{\substack{k,n_1,n_2 \\ n_1 \neq n_2}} \ \frac{1}{k^2}\ N_{n_1,n_2}(q_1^{n_1} q_2^{n_2})^k \ .$$

The first few coefficients N_{n_1,n_2} for small n_i, including the contributions from $n_1 = n_2$, are listed in table 1 below.

$n_2 \backslash n_1$	0	1	2	3	4	5
0	0	−320	13280	−1088960	119783040	−15440622400
1	20	1600	−116560	12805120	−1766329640	274446919680
2	0	2040	679600	−85115360	13829775520	−2525156504560
3	0	−1460	1064180	530848000	−83363259240	16655092486480
4	0	520	−1497840	887761280	541074408000	−95968626498800
5	0	−80	1561100	−1582620980	931836819440	639660032468000
6	0	0	−1152600	2396807000	−1864913831600	1118938442641400
7	0	0	580500	−2923203580	3412016521660	−2393966418927980
8	0	0	−190760	2799233200	−5381605498560	4899971282565360
9	0	0	37180	−2078012020	7127102031000	−9026682030832180
10	0	0	−3280	1179935280	−7837064629760	14557931269209000
11	0	0	0	−502743680	7104809591780	−20307910970428360
12	0	0	0	155860160	−5277064316000	24340277955510560
13	0	0	0	−33298600	3187587322380	−24957649473175420
14	0	0	0	4400680	−1549998228000	21814546476229120
15	0	0	0	−272240	597782974040	−16191876966658500
16	0	0	0	0	−178806134240	10157784412551120
17	0	0	0	0	40049955420	−5351974901676280

Table 1: Predictions for Ooguri–Vafa invariants.

According to the general philosophy of the Hodge theoretic mirror map described in the previous sections, the double logarithmic solutions represent the generating function of holomorphic discs ending on the A-brane L. In the basis of sect. 4 we find

$$F_t = \mathcal{T}^{(1)} + \mathcal{T}^{(2)} = \mathcal{T}, \qquad W = \mathcal{T}^{(1)} .$$

Assuming that the normalization argument leading to (5.3) is correct, the numbers N_{n_1,n_2} of table 1 are genuine Ooguri-Vafa invariants for the A-brane geometry predicted by mirror symmetry. It would be interesting to justify the above arguments and the prediction for the disc invariants in table 1 by an independent computation. Further evidence for the above results will be given in sect. 7 and app. A, by deriving the same result from the afore mentioned duality to Calabi–Yau four-folds.

6. Relation to CFT correlators

The relevant closed-string observables in the BRST cohomology of the topological B-model of a Calabi-Yau manifold are locally given by [48]

$$\phi^{(p)} \;=\; \phi^{(p)}{}_{\bar\imath_1\cdots\bar\imath_p}^{j_1\cdots j_p}\eta^{\bar\imath_1}\cdots\eta^{\bar\imath_p}\theta_{j_1}\cdots\theta_{j_p} \ , \tag{6.1}$$

where the world-sheet fermions, $\eta^{\bar\imath}=\psi_+^{\bar\imath}+\psi_-^{\bar\imath}$ and $\theta_i=g_{i\bar\jmath}\left(\psi_+^{\bar\jmath}-\psi_-^{\bar\jmath}\right)$, are sections of the pullbacks of the anti-holomorphic tangent bundle and the holomorphic cotangent bundle of target-space Calabi-Yau manifold. For the Calabi-Yau three-fold Z^* these observables $\phi^{(p)}$ are identified geometrically with representatives in the sheaf cohomology groups

$$\phi^{(p)}\in H^p(Z^*,\Lambda^pTZ^*)\simeq H^{(3-p,p)}(Z^*) \ , \qquad p=0,1,2,3 \ . \tag{6.2}$$

The last identification is due to the contraction with the unique holomorphic (3,0) form of the Calabi-Yau three-fold Z^*. The integer p represents the left and right $U(1)$ charge of the bulk observable $\phi^{(p)}$.

The local open-string observables for a worldsheet with B-type boundary are analogously given by

$$\hat\phi^{(p+q)} \;=\; \hat\phi^{(p+q)}{}_{\bar\imath_1\cdots\bar\imath_p}^{j_1\cdots j_q}\eta^{\bar\imath_1}\cdots\eta^{\bar\imath_p}\theta_{j_1}\cdots\theta_{j_q} \ . \tag{6.3}$$

In the absence of a background gauge field on the worldvolume of the brane the fermionic modes θ_j vanish along Neumann directions whereas the fermionic modes $\eta^{\bar\imath}$ vanish along Dirichlet directions on the boundary of the worldsheet [49]. Hence, locally we view the fermionic modes θ_j as sections of the normal bundle and the fermionic modes $\eta^{\bar\imath}$ as sections of the anti-holomorphic cotangent bundle of the brane. With background fluxes on the brane worldvolume the boundary conditions become twisted and obey [2]

$$\theta_i=F_{i\bar\jmath}\eta^{\bar\jmath} \ . \tag{6.4}$$

In ref. [14] it is explicitly demonstrated that the observables (6.3) in the BRST cohomology of the open-string sector for a brane E arise geometrically as elements of the extension groups

$$\hat\phi^{(p+q)}\in \mathrm{Ext}^{p+q}(E,E) \ , \qquad p+q=0,1,2,3 \ . \tag{6.5}$$

In the present context, the integer $p+q$ is equal to the total $U(1)$ charge of the open-string observable $\hat\phi^{(p+q)}$.

Deformations of the topological B-model are generated by the marginal operators, which correspond to BRST observables with $U(1)$ charge one, and hence they appear in

the cohomology groups $H^{(2,1)}(Z^*)$ and $\text{Ext}^1(E,E)$ for the closed and open deformations, respectively.

In order to make contact with the Hodge filtration of $H^3(Z^*,D)$ we interpret the divisor D of the Calabi-Yau three-fold Z^* as the internal worldvolume of a B-type brane. For a divisor the extension groups (6.5) simplify [14], and in particular $\text{Ext}^1(D,D)$ reduces to $H^0(D,ND) \simeq H^{(2,0)}(D)$, where the last identification results again from the contraction with the holomorphic (3,0) form. For our particular example the cohomology groups $H^{(2,1)}(Z^*)$ and $H^{(2,0)}(D_1)$ are both one-dimensional and therefore are generated by the closed- and open-string marginal operators ϕ and $\hat{\phi}$

$$\phi^{(1)} \in H^{(2,1)}(Z^*) \subset F^2/F^3 \ , \qquad \hat{\phi}^{(1)} \in H^{(2,0)}(D_1) \subset F^2/F^3 \ .$$

Due to the identification $F^2/F^3 = H^{(2,1)}(Z^*,D_1) \simeq H^{(2,1)}(Z^*) \oplus H^{(2,0)}(D_1)$ we observe that the infinitesimal deformations $\nabla_t \alpha_1^{(0)} \sim \phi^{(1)}$ and $\nabla_{\hat{t}} \alpha_1^{(0)} \sim \hat{\phi}^{(1)}$ in eq. (4.3) precisely agree with the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. As a consequence the discussed Picard-Fuchs equations, governing the Hodge filtration F^p, describe indeed the deformation space associated to the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$.

In the presence of B-type boundaries infinitesimal deformations are generically obstructed at higher order. These obstructions are encoded in the moduli-dependent superpotential generated by disc correlators with insertions of bulk and boundary marginal operators [50,51,52,53]. The relevant disc correlators arise from non-trivial ring relations involving marginal operators and the (unique) boundary top element $\hat{\phi}^{(3)} \in \text{Ext}^3(D,D)$. Hence the superpotential is extracted by identifying the element $\hat{\phi}^{(3)}$ in the relative cohomology group $H^3(Z^*,D)$. For the family of hypersurfaces D the extension group $\text{Ext}^3(D,D)$ becomes [14]

$$\hat{\phi}^{(3)} \in \text{Ext}^3(D,D) \simeq H^2(D,ND) \simeq H^{(2,2)}(D) \ ,$$

where locally $\hat{\phi}^{(3)} = \hat{\phi}^{(3)}{}_{\bar{\imath}\bar{\jmath}}^{k} \eta^{\bar{\imath}} \eta^{\bar{\jmath}} \theta_k$. It is obvious that the cohomology group $H^{(2,2)}(D)$ does not appear in the filtration F^p of the relative cohomology group $H^3(Z^*,D)$. On the other hand the variation of mixed Hodge structure encodes by construction the ring relations of the observables generated by the marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. Therefore we conclude that these marginal operators do not generate the boundary-boundary top element $\hat{\phi}^{(3)}$. Thus the analyzed deformation problem is unobstructed and does not give rise to a non-vanishing superpotential.

From a physics point of view the family of divisors D describes a family of holomorphic hypersurfaces, which all give rise to supersymmetric B-brane configurations, and hence we should not expect any obstructions resulting in a superpotential.

However, the result of the above analysis drastically changes as we add a $D5$-brane charge on 2-cycles in D, e.g. by adding non-trivial background fluxes on the worldvolume of the B-type brane. From a space-time perspective [54,55], we expect the appearance of F-terms precisely for those two-form background fluxes, whose field strength takes values in the variable cohomology of the hypersurface D

$$F \in \text{coker } \left(H^2(Z^*, \mathbf{Z}) \to H^2(D, \mathbf{Z})\right) . \tag{6.6}$$

These fluxes induce a macroscopic superpotential [54,55,56,57]

$$W = \int_D F \wedge \omega = \int_\Gamma F \wedge \Omega, \tag{6.7}$$

where $\omega \in H^{2,0}(D)$ is obtained by contracting the bulk (3,0) form Ω with a section of the normal bundle to D. The second expression, derived in a more general context in ref. [57], is equivalent to the first one for an appropriate choice of 5-chain with boundary D.

In the microscopic worldsheet description the worldvolume flux $F_{i\bar{\jmath}}$ yields twisted boundary conditions (6.4), and the fermionic modes θ_i of the open-string observables (6.3) are in general no longer sections of the normal bundle ND. Instead they should be viewed as appropriate section in the restricted tangent bundle, $TZ^*|_D$ [14]. As a consequence we can trade (without changing the $U(1)$ charge) fermionic modes $\eta^{\bar{\jmath}}$ with appropriate fermionic modes θ_i. As a result the boundary top element $\hat{\phi}^{(3)}$ can now be associated with an element in the variable two-form cohomology

$$\text{Ext}^3(D,D) \ni \hat{\phi}^{(3)k}_{\bar{\imath}\bar{\jmath}} \eta^{\bar{\imath}} \eta^{\bar{\jmath}} \theta_k \overset{F_{i\bar{\jmath}}}{\longleftrightarrow} \hat{\phi}^{(3)jk}_{\bar{\imath}} \eta^{\bar{\imath}} \theta_j \theta_k \overset{\Omega_{ijk}}{\longleftrightarrow} \hat{\phi}^{(3)}_{i\bar{\jmath}}\, dx^i \wedge dx^{\bar{\jmath}} \in \text{coker } \left(H^2(Z^*) \to H^2(D)\right) . \tag{6.8}$$

Thus in the presence of worldvolume background fluxes the boundary top element $\hat{\phi}^{(3)}$ *does* correspond to an element in the Hodge structure filtration F^p, and the superpotential is described by a solution of the Picard-Fuchs equations. In this way the a priori unobstructed deformation problem of divisors D in the Calabi-Yau three-fold is capable to describe superpotentials associated to $D5$-brane charges in $H_2(D)$ [6,7,9].

On the other hand, since the discussed F-term fluxes (6.6) are elements of the variable cohomology of the hypersurface D, *i.e.* the field strength of the fluxes can be extended to exact two forms in the ambient Calabi-Yau space, they do not modify the $D5$-brane K-theory charges. Therefore if a suitable $D5$-brane interpretation is applicable the flux-induced superpotentials describe domain-wall tensions between pairs of $D5$-branes, which wrap homologicaly equivalent two cycles.

When written in the flat coordinates $t_a = \frac{1}{2\pi i}\ln(q_a)$ in (4.7), the Gauss-Manin connection on the total cohomology space takes the form:

$$
\begin{pmatrix} \nabla\alpha_1^{(0)} \\ \nabla\alpha_2^{(1)} \\ \nabla\alpha_3^{(1)} \\ \nabla\alpha_4^{(2)} \\ \nabla\alpha_5^{(2)} \\ \nabla\alpha_6^{(3)} \\ \nabla\alpha_7^{(3)} \end{pmatrix} = \sum_b \begin{pmatrix} 0 & C_b^{(0)}(q_a)\frac{dq_b}{q_b} & 0 & 0 \\ 0 & 0 & C_b^{(1)}(q_a)\frac{dq_b}{q_b} & 0 \\ 0 & 0 & 0 & C_b^{(2)}(q_a)\frac{dq_b}{q_b} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1^{(0)} \\ \alpha_2^{(1)} \\ \alpha_3^{(1)} \\ \alpha_4^{(2)} \\ \alpha_5^{(2)} \\ \alpha_6^{(3)} \\ \alpha_7^{(3)} \end{pmatrix} . \tag{6.9}
$$

The most notable difference to the closed-string case (cf. eq.(4.8)) is that, whereas the matrix $C_b^{(0)}$ still is of the canonical form $(C_b^{(0)})_1{}^l = \delta_{bl}$, the matrices $C_b^{(q)}$ are now both moduli dependent for $q = 1, 2$:

$$
(C_t^{(1)}) = \begin{pmatrix} C & W_{,tt} \\ 0 & W_{,t\hat{t}} \end{pmatrix}, \ (C_{\hat{t}}^{(1)}) = \begin{pmatrix} 0 & W_{,t\hat{t}} \\ 0 & W_{,\hat{t}\hat{t}} \end{pmatrix}, \ (C_t^{(2)}) = \begin{pmatrix} -1 & \mu_{,t} \\ 0 & \rho_{,t} \end{pmatrix}, \ (C_{\hat{t}}^{(2)}) = \begin{pmatrix} 0 & \mu_{,\hat{t}} \\ 0 & \rho_{,\hat{t}} \end{pmatrix},
$$

In correspondence with the closed-string sector it is tempting to interprete the $d_{q-1} \times d_q$ matrices $C_b^{(q)}$ as the structure constants of a ring of open and closed chiral operators

$$
\phi_b^{(1)} \cdot \phi_k^{(q)} \ \stackrel{?}{=} \ (C_b^{(q)})_k{}^l \phi_l^{(q+1)} \ ,
$$

as described in [6,15]. A rigorous CFT derivation of such a relation is non-trivial, as the Hodge variation mixes bulk and boundary operators and describes the bulk-boundary ring in the sense of ref. [14], about which little is known in the context of topological strings (see however refs. [58,59]). A related complication is the need of a topological metric on the space of closed and open BRST states that mixes contributions at different order of the string coupling. The most direct way to connect the closed-string periods with a CFT quantity is the interpretation as overlap functions between boundary states and chiral operators [11], and it is likely that a similar idea can be applied to the entries of the relative period matrix. It would be interesting to make this precise. It would also be interesting to understand more generally the relation of the above concepts to the CFT results obtained from matrix factorizations in refs. [60,61,62,63].

7. F-theory four-folds and effective $\mathcal{N} = 1$ supergravity

The four-dimensional effective action for the brane compactification (Z, L), or its mirror (Z^*, E), should be described by a general $\mathcal{N} = 1$ supergravity theory, which depends on the Kähler potential and the superpotential through the function [64]

$$
\mathcal{G} = K(z, \hat{z}; \bar{z}, \bar{\hat{z}}) + \ln W(z, \hat{z}) + \ln \bar{W}(\bar{z}, \bar{\hat{z}}) . \tag{7.1}
$$

Here $(z, \hat{z})$ denote again local complex coordinates on the bundle $\mathcal{M} \to \mathcal{M}_{CS}$. Similarly the construction of ref. [9] associates to the mirror pair of brane compactifications (Z, L) and (Z^*, E) a dual four-fold compactification with the same supersymmetry. Some details of this duality, including a lift to a full F-theory compactification, will be explained below. Subsequently we compare the effective couplings in these two descriptions and show that they lead to a consistent proposal for the $\mathcal{N} = 1$ supergravity function $\mathcal{G}(z, \hat{z}; \bar{z}, \bar{\hat{z}})$.

7.1. Four-fold dualities and F-theory

The construction of refs. [8,9] associates a toric four-fold polyhedron Δ to a toric brane configuration defined as in ref. [3]. For the quintic example, Δ is defined in (3.21) and has the property that the toric GKZ system associated with Δ reproduces the GKZ system for the relative cohomology group for the brane compactification (Z^*, E) via eq.(3.22). In the following we describe some details of the duality map on the target and moduli spaces.

First note that the polyhedron Δ and its dual polyhedron Δ^* actually define a mirror pair of CY four-folds X and X^*. As is clear from the derivation of the Picard-Fuchs equations in sect. 3, the relative periods of the brane compactification (Z^*, E) are identified with the periods of holomorphic $(4, 0)$ form on the four-fold X^*. It follows that the complex structure deformations of the brane compactification (Z^*, E) map to the complex structure of the four-fold X^*. Adding mirror symmetry, one obtains the following relation between the different compactifications:

$$\begin{array}{ccc} \begin{matrix}(Z, L)\\ (\mathrm{A-branes})\end{matrix} & \xleftrightarrow[\ symmetry\ ]{mirror} & \begin{matrix}(Z^*, E)\\ (\mathrm{B-branes})\end{matrix} \\ {\scriptstyle 4f}\big\downarrow{\scriptstyle dual} & & {\scriptstyle 4f}\big\downarrow{\scriptstyle dual} \\ X & \xleftrightarrow[\quad symmetry\quad]{mirror} & X^* \end{array} \tag{7.2}$$

The vertical maps in this diagram preserve, whereas the horizontal maps exchange, the notion of complex and Kähler moduli.

The mirror pair (X, X^*) of four-folds constructed in this way has a very special geometric structure that reflects the mirror symmetry between A-type and B-type branes on the mirror pair (Z, Z^*) of CY three-folds. Whereas the correspondence between *moduli spaces* is manifest on the B-type side as the relation between the periods of (Z^*, E)

and X^*,[22] there is a simple correspondence between the *target spaces* on the A-type side. Namely, the mirror four-fold X is a fibration over the complex plane

$$\begin{array}{ccc} Z & \longrightarrow & X \\ & & \big\downarrow \pi \\ & & \mathbf{C} \end{array} \tag{7.3}$$

with generic fiber a CY three-fold of type Z and a degenerate central fiber at the origin specified by the toric polyhedron constructed in refs. [8,9]. Thus the dual four-fold X^* that captures the relative periods of the brane compactification (Z^*, E) is effectively constructed by fibering the CY three-fold Z for the A-branes over $\mathbf{C}$ and then taking the (four-fold) mirror of the fibration X obtained in this way.

The mirror pair (X, X^*) of non-compact CY four-folds can be related to a honest four-dimensional F-theory compactification by a simple $\mathbf{P}^1$ compactification of the non-compact base of X. In this way one obtains a mirror pair of compact CY four-folds (X_c, X_c^*), where X_c^* is the four-fold for F-theory compactification.

$$\begin{array}{ccccc} Z & \longrightarrow & X_c & \underset{symmetry}{\overset{mirror}{\longleftrightarrow}} & X_c^* \quad (\mathrm{F-theory}) \\ & & \big\downarrow \pi & & \\ & & \mathbf{P}^1 & & \end{array} \tag{7.4}$$

An important point is to identify to the image of the large base limit of X_c in the moduli space of the F-theory compactification on X_c^*, which can be deduced from the mirror map and the methods of refs. [65,66]. The result is that the large volume limit $\mathrm{Im}\, S = Vol(\mathbf{P}^1) \to \infty$ maps under mirror symmetry to a weak coupling limit $g_s \to 0$

$$Vol(\mathbf{P}^1) \sim 1/g_s \ \to \infty \ . \tag{7.5}$$

Thus the pair of four-folds (X, X^*) is recovered in the decompactification/weak-coupling limit and the diagram (7.2) is completed downwards to

$$\begin{array}{ccc} X & \underset{symmetry}{\overset{mirror}{\longleftrightarrow}} & X^* \\ {\scriptstyle Vol(\mathbf{P}^1)\to\infty}\big\uparrow & & \big\uparrow{\scriptstyle g_s\to 0} \\ X_c & \underset{symmetry}{\overset{mirror}{\longleftrightarrow}} & X_c^* \end{array} \tag{7.6}$$

[22] An explicit match of period integrals for a class of examples can be found in ref. [8].

On the one hand, the details of the $\mathbf{P}^1$ compactification determine the subleading corrections in g_s but become irrelevant in the decompactification/decoupling limit. On the other hand it is worth stressing, that the "duality" between non-compact Calabi–Yau four-folds (X, X^*) and the mirror pair of brane compactifications (Z, L) and (Z^*, E), which underlies the superpotential computation, represents the *strict* limit $g_s = 0$, where most of the degrees of freedom decouple from the superpotential sector. A true duality can exist, if at all, only at the level of the lower row of the above diagram.

In the concrete example of sect. 4, the $\mathbf{P}^1$ compactification can be obtained by adding the extra vertex

$$\tilde{\nu}_8 = (1, 0, 0, 0, -1) \tag{7.7}$$

to the toric polyhedron (3.21) of the non-compact CY four-fold X. This defines a compact Calabi–Yau four-fold X_c. The Kähler modulus S of the compact $\mathbf{P}^1$ base is described by the charge vector

$$\tilde{l}^3 = (0; -2, 0, 0, 0, 0, 0, 1, 1).$$

The mirror manifold X_c^* is elliptically fibered and defines an F-theory compactification that will be used to compute the effective couplings in the effective four-dimensional $\mathcal{N} = 1$ supergravity theory in the following section.

7.2. *Effective $\mathcal{N} = 1$ supergravity*

According to the above discussion the $\mathcal{N} = 1$ superpotential appearing in the $\mathcal{N} = 1$ supergravity function (7.1) is

$$W(z, \hat{z}) = \sum_{\Sigma} \underline{N}_{\Sigma} \int_{\gamma_{\Sigma}} \underline{\Omega}^{(3,0)}, \qquad \gamma_{\Sigma} \in H_3(Z^*, D) \tag{7.8}$$

if we consider the brane compactification (Z^*, E) as in refs. [6,7,9], or, alternatively

$$W(z, \hat{z}, S) = \sum_{\Sigma} N_{\Sigma} \int_{\gamma_{\Sigma}} \Omega^{(4,0)}, \qquad \gamma_{\Sigma} \in H_4(X^*) \ , \tag{7.9}$$

for the F-theory compactification on the dual four-fold X_c^*.[23] As discussed above, the difference between the four-fold periods and the (relative) three-fold periods are subleading

[23] See also refs. [67,68] for an early discussion of four-fold periods and superpotentials in lower dimensions.

corrections in small g_s; see app. A for some details of the computation and a precise match between the periods in the example.

As for the Kähler potential, consider first the $\mathcal{N}=2$ Kähler potential on the base of the fibration $\mathcal{M}$, that is on the complex structure moduli space $\mathcal{M}_{CS}$ for the string compactification on Z^* without branes. This is given by [69]

$$K_{CS}(z;\bar{z}) = -\ln Y_{CS}\ ,\qquad Y_{CS} = -i\int_{Z^*}\Omega\wedge\bar{\Omega} = -i\sum_{\gamma_\Sigma\in H_3(Z^*)}\Pi^\Sigma(z)\,\eta_{\Sigma\Lambda}\,\bar{\Pi}^\Lambda(\bar{z})\ .$$

Here $\Sigma,\Lambda = 1,...,h^3(Z^*)$ and $\eta_{\Sigma\Lambda}$ is the symplectic intersection matrix on $H_3(Z^*,\mathbf{Z})$, which represents the constant, topological metric on the space of groundstates in the SCFT [39]. Restricting the sum in eq.(7.8) to the "flux" superpotential, that is to the absolute cohomology $H_3(Z^*)$, one obtains a function

$$\mathcal{G}_{CS} = K_{CS}(z;\bar{z}) + \ln W_{CS}(z) + \ln\bar{W}_{CS}(\bar{z}),\qquad W_{CS}(z) = \sum_{\gamma_\Sigma\in H^3(Z^*)} N_\Sigma\Pi_\Sigma(z)\ ,\quad (7.10)$$

that depends only on the closed string moduli and is invariant under Kähler transformations generated by rescalings of the holomorphic (3,0) form, $\Omega\to e^f\Omega$.

We will now give two independent arguments, that the $\mathcal{N}=1$ Kähler potential on the full $\mathcal{N}=1$ deformation space $\mathcal{M}$ can be written, to leading order in g_s, as

$$K(z,\hat{z};\bar{z},\bar{\hat{z}}) = -\ln Y\ ,\qquad Y = -i\sum_{\gamma_\Sigma\in H_3(Z^*,D)}\underline{\Pi}^\Sigma(z,\hat{z})\,\underline{\eta}_{\Sigma\Lambda}\,\underline{\bar{\Pi}}^\Lambda(\bar{z},\bar{\hat{z}})\ .\qquad (7.11)$$

with a pairing matrix $(\underline{\eta})$ defined below. Indeed this ansatz is a natural guess in view of the extension of the summation from $H^3(Z^*)$ to $H^3(Z^*,D)$ in the $\mathcal{N}=1$ superpotential (7.8) and defines an $\mathcal{N}=1$ supergravity function $\mathcal{G}(z,\hat{z};\bar{z},\bar{\hat{z}})$ which is invariant under Kähler transformations generated by rescalings of the relative (3,0) form

$$\underline{\Omega}\to e^f\underline{\Omega},\qquad \Pi_\Sigma\to e^f\underline{\Pi}_\Sigma\ .$$

Note that since the Kähler metric for the closed and open string deformations arises from different world-sheet topologies, the pairing $(\underline{\eta})$ on $H_3(Z^*,D)$ necessarily mixes terms of different order in the string coupling g_s.

The first argument comes from the results of ref. [13] on the effective space-time action for orientifold compactifications of $D7$-branes. It has been shown there, that the Kähler metric obtained by dimensional reduction of a $D7$-brane worldvolume wrapping

the orientifold plane $D^\sharp$ in an orientifold $Z^\sharp$ is consistent, at first order in the brane deformation, with a Kähler potential $K = -\ln Y_{OF}$ with

$$Y_{OF} = -i \int_{Z^\sharp} (P^{(3)}\underline{\Omega}) \wedge (P^{(3)}\underline{\bar{\Omega}}) + \tilde{g} \int_{D^\sharp} (P^{(2)}\underline{\Omega}) \wedge (P^{(2)}\underline{\bar{\Omega}}) \ . \tag{7.12}$$

Here $P^{(3)}$ and $P^{(2)}$ are projection operators onto the two summands in eq. (2.3), and $\tilde{g}$ is g_s times a constant. This is of the form (7.11) with the pairing matrix

$$(\underline{\eta}) = \begin{pmatrix} \eta_{Z^*} & 0 \\ 0 & i\tilde{g}\,\tilde{\eta}_{D^\sharp} \end{pmatrix} , \tag{7.13}$$

where $\tilde{\eta}_{D^\sharp}$ is the (symmetric) intersection matrix on $H^2_{var}(D^\sharp)$.

The second argument is obtained by computing the Kähler potential of the dual four-fold X^*_c, which is of a similar form as eq.(7.11) [70]:

$$K(z,\hat{z},S;\bar{z},\bar{\hat{z}},\bar{S}) = -\ln Y \ , \qquad Y = \sum_{\gamma_\Sigma \in H_4(X^*)} \Pi^\Sigma(z,\hat{z},S)\,\eta_{\Sigma\Lambda}\,\bar{\Pi}^\Lambda(\bar{z},\bar{\hat{z}},\bar{S}) \ . \tag{7.14}$$

Here η denotes the topological intersection matrix on $H_4(X^*_c)$ and S is the afore mentioned extra modulus in the compact manifold. An explicit computation[24] in the weak coupling limit $\mathrm{Im}\,S \to \infty$ then shows that the four-fold Kähler potential can be rewritten, to leading order in g_s, as the sum of two terms, corresponding to a split (7.13) with the two blocks given by the symplectic form η_{Z^*} on $H_3(Z^*,\mathbf{Z})$ and $\tilde{\eta}_D$ the intersection matrix on the hypersurface D. Thus, to leading order in g_s, the F-theory result is in perfect agreement with the local orientifold result of ref. [13].

The role of the intersection matrix $\tilde{\eta}_D$ as defining a topological metric in the open-string sector is natural in view of the localization of the open-string degrees of freedom on D. In the quintic example, D describes the K3 geometry that captures the variation of the chain integrals with the open-string deformations, as described below (3.19), and $\tilde{\eta}_D$ is simply the K3 intersection matrix $\hat{\eta}$ discussed below (4.13). Explicitly, the resulting Kähler potential for the four-fold X^*_c defined by eqs.(3.21),(7.7), reads, to leading order in small g_s and in an expansion near the large complex structure point,

$$K = -\ln(\tilde{g}^{-1}\,\tilde{Y}), \qquad \tilde{Y} = \frac{5i}{6}(t-\bar{t})^3 + \tilde{g}\big(-\frac{1}{6}(t_1-\bar{t_1})^4 + \frac{5}{12}(t-\bar{t})^4\big) + \mathcal{O}(|t|^2) \ , \tag{7.15}$$

where (t,t_1) are the flat coordinates of eq.(5.1) and $\tilde{g}^{-1} = 2\,\mathrm{Im}\,S$. The second summand in the order $\tilde{g}^1$ term is a correction predicted by the dual F-theory four-fold, which is not captured by the dimensional reduction, eq. (7.12). By mirror symmetry, eq. (7.15) then represents a prediction for the Kähler metric on the deformation space of A-type branes on the quintic Z. It would be interesting to understand the relation of the above proposal to the metric described by Hitchin in ref. [71].

[24] See app. A. for details.

8. Summary and outlook

In this work we analyzed the deformation problem of certain families of toric D-branes in compact Calabi-Yau three-folds, defined along the lines of ref.[3]. This is achieved by studying the variation of Hodge structure as described by the periods of the holomorphic three-form of the Calabi-Yau manifold while keeping track of the boundary contributions relative to a family of four-cycles describing the B-brane geometry. We demonstrate our techniques with a specific B-brane configuration in the mirror quintic. Although this geometry serves as a guiding example throughout the paper, we present also a general toric description of the generalized hypergeometric systems governing the toric brane configurations, for which our discussion applies.

We find that, similarly to the well-studied deformation problem in the pure closed-string sector, the notions of flatness and integrability of the Gauss-Manin connection continue to make sense on the open-closed deformation space $\mathcal{M}$ of the family and lead to sensible results for open string enumerative invariants. Amongst others, the Gauss-Manin connection in flat coordinates displays an interesting ring structure on the infinitesimal deformations in F^2, which is compatible with CFT expectations and gives evidence for the existence of an A-model quantum product defined by the Ooguri-Vafa invariants. Other hints in this direction are the integrability condition and the meaningful definition of the mirror map (4.7) via a flatness condition.

For geometries with a single open string modulus the integrability conditions imply that the relative period matrices and the Gauss-Manin connection matrices can all be expressed in terms of functional relations involving only the holomorphic prepotential $\mathcal{F}$ and one additional holomorphic function W.[25]

The analyzed open-closed deformation problem can also be related to CFT correlators. We explained how an a priori unobstructed deformation problem of B-branes wrapping a holomorphic family of four-cycles describes an obstructed deformation problem after turning on $D5$-brane charges. In particular this effect can be described in the CFT by the change of boundary conditions induced by non-trivial fluxes on the worldvolume of the B-brane. The afore mentioned holomorphic function W in the Gauss-Manin connection then turns into a superpotential encoding these obstructions.

By mirror symmetry our analysis carries over to the quantum integrable structure of the obstructed deformation space of the mirror A-branes in the mirror three-fold. For our explicit example we obtain predictions for the Ooguri-Vafa invariants of the open-closed

[25] This can be generalized to cases with several open-string moduli studied in [40].

deformation space that satisfy the expected integrality constraints and further consistency conditions.

Turning to the effective four-dimensional $\mathcal{N}=1$ supergravity theory one observes that the open-closed deformation space $\mathcal{M}$ is a fibration $\pi:\ \mathcal{M}\to\mathcal{M}_{CS}$ over the complex structure moduli and defines a Kähler manifold that is not of the most general form allowed by supergravity, but has a restricted "$\mathcal{N}=1$ special geometry". By constructing a class of dual Calabi–Yau four-folds for F-theory compactification we derived an expression for the effective $\mathcal{N}=1$ Kähler potential and the superpotential on $\mathcal{M}$ in terms of period integrals. The F-theory compactification on the dual four-fold provides a global embedding of the B-brane geometry on the three-fold and reduces to a local description in the decompactification/weak-coupling limit (7.5). The effective description obtained in this limit is in good agreement with the results obtained for $D7$-branes on orientifolds in the existing literature.

However, the B-model results of this paper, obtained predominantly from a Hodge theoretic approach, raise also a number of unanswered questions. The first is about the meaning of mirror symmetry between open-closed deformation spaces in the presence of a non-trivial superpotential, which requires some sort of off-shell concept of mirror symmetry. As discussed above, a heuristic ansatz might be to define $\mathcal{M}$ first as the deformation space of an unobstructed family and then add in obstructions as a sort of perturbation, here represented by $D5$-brane charges. However, we feel that there should be a more fundamental answer to this important issue.

Another set of urgent questions concerns the A-model interpretation, such as a proper formulation of an A-model quantum ring that matches the ring structure observed on the B-model side and should include the Ooguri-Vafa invariants and Floer (co-)homology as essential ingredients. Similarly one would like to have a more explicit description of the target space geometry of the A-branes. We hope to come back to these questions in the future.

The Hodge theoretic approach to open-string mirror symmetry is somewhat complementary to other approaches to open-string mirror symmetry in the recent literature. For compact manifolds these include the study of critical superpotentials without open moduli in refs. [4,25,72,73,74] and the computation of effective superpotentials in terms of matrix factorizations and related techniques [60,61,62,63,51,52,75]. It is likely that the "off-shell" description of the open-closed deformation space advocated for in this paper is useful to study the interesting phenomena of phase transitions of domain walls [74,43],

which resemble similar phase transitions in the closed-string sector, such as conifolds and flops.

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Appendix A. W and K from four-folds in the quintic example

In the following, we hand in some technical details of the computation of the superpotential (7.9) and the Kähler potential (7.15) from the F-theory four-fold X_c presented in sect. 7 and substantiate some of the claims made there. This includes an explicit illustration of the relation between the four-fold flux superpotential (7.9) and the three-fold superpotential (7.8) from RR and NS fluxes in the weak coupling limit as well as the derivation of the Kähler potential (7.15). To avoid excessive repetitions, we refer to refs. [70,68,76] for the details on mirror symmetry for Calabi–Yau four-folds and to ref. [65] for the application of toric geometry to analyze geometry of the relevant F-theory four-folds. In the context of open-closed superpotentials the method described below has been previously used to compute the superpotentials for several other examples in ref. [9].

The A-model four-fold X_c is defined by the polyhedron Δ with vertices given in (3.21), (7.7). We consider a phase of the Kähler cone described by the charge vectors given in the text, which we reproduce here for convenience:

$$\begin{aligned}
\tilde{l}^1 &= (\ \ -4 \quad 0 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ -1 \quad 1 \ \ 0)\ ,\\
\tilde{l}^2 &= (\ \ -1 \quad 1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \quad 1 \ \ -1 \ \ 0)\ ,\\
\tilde{l}^3 &= (\quad 0 \ \ -2 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \quad 0 \quad 1 \ \ 1)\ .
\end{aligned} \tag{A.1}$$

The Kähler form is $J = \sum_a t_a J_a$, where J_a, $a = 1, 2, 3$ denotes the basis of $H^{1,1}(X_c)$ dual to the Mori cone defined by (A.1). The basic topological data are the intersections

$$K_{abcd} = \int_{X_c} J_a \wedge J_b \wedge J_c \wedge J_d\ ,$$

which can be concisely summarized in terms of the generating function[26]

$$\begin{aligned}
\mathcal{F}_4 &= \frac{1}{4!}\int_{X_c} J^4 = \frac{1}{4!}\sum_{a,b,c,d} K_{\alpha\beta\gamma\delta} t^\alpha t^\beta t^\gamma t^\delta \\
&= \frac{1}{4}t_1^4 + \frac{5}{3}t_1^3(t_2 + \frac{1}{2}t_3) + \frac{5}{2}t_1^2 t_2(t_2+t_3) + \frac{5}{3}t_1 t_2^2(t_2 + \frac{3}{2}t_3) + \frac{5}{12}t_2^3(t_2 + 2t_3)\ .
\end{aligned} \tag{A.2}$$

[26] Details on the computation of the following data from toric geometry and many sample computations for three-fold fibered four-folds can be found in [68,76].

From these intersections one obtains the following topological invariants of X_c[27]

$$
\begin{aligned}
\chi &= \int_{X_c} c_4 = 1860 = 12 \bmod 24\,, \qquad\qquad R_4 = \int_{X_c} c_2^2 = 1100, \\
R_2 &= \int_{X_c} J\wedge J\wedge c_2 = 90J_1^2 + 110J_2^2 + 220J_1J_2 + 100J_3(J_1+J_2)\,, \\
R_3 &= \int_{X_c} J\wedge c_3 = -330J_1 - 410J_2 - 200J_3\,,
\end{aligned}
\tag{A.3}
$$

where c_k denote the Chern classes of X_c. The independent Hodge numbers $h^{1,1} = 3$, $h^{1,2} = 0$, $h^{1,3} = 299$ can be read off from the toric polyhedra [68,76] and fix $h^{2,2} = 12 + \frac{2}{3}\chi + 2h^{1,2} = 1252$; we refer to [77] for more details and the meaning of the mod 24 condition on χ.

After a linear change of coordinates, the form of the generating function $\mathcal{F}_4$ simplifies to

$$
\mathcal{F}_4 = S\ \frac{5}{6}t^3\ +\ \big(\frac{5}{12}t^4 - \frac{1}{6}t_1^4\big)\ , \tag{A.4}
$$

where $t_3 = S$ is the Kähler modulus of the $\mathbf{P}^1$ base, $t_1+t_2 = t$ is the modulus of the generic quintic three-fold fiber and $t_2 = \hat{t}$ will be related to the open string modulus denoted by the same letter in sect. 4. Note that the fibration structure becomes explicit in the coordinates (S, t) and that the leading term in large S contains the intersection form on the quintic fiber Z. To simplify some of the following expressions, we use the linear combinations

$$
t_1' = t = t_1 + t_2, \qquad t_2' = \hat{t} - t = -t_1, \qquad t_3' = S = t_3\ .
$$

To compute the superpotential and the Kähler potential we have to determine the integral periods of the elements of the vertical subspace $H^{2q}_{ver}(X_c)$ generated by wedge products of the elements $J_a \in H^2(X_c)$ over topological cycles in $H_{2q}(X_c, \mathbf{Z})$, for $q = 0, ..., 4$. Except for $q = 2$, there is a canonical basis for $H_{2q}(X_c, \mathbf{Z})$ given by the class of a point, the class of X_c, the divisors dual to the generators J_a and the curves dual to these divisors, respectively.[28] In an expansion near the large volume point $\mathrm{Im}\, t_a \to \infty\ \forall a$, the leading

[27] Contrary to first appearances, our choice of X_c is not at all related to a preference for a particular soccer club.

[28] To be precise, some of these basis elements may actually be integral multiples of the generators of $H_{2q}(X_c, \mathbf{Z})$ on the hypersurface.

part of the periods, denoted by $\tilde{\Pi}_{q,\cdot}$, is, up to a sign, the Kähler volume of the cycle as measured by the volume form $\frac{1}{k!}J^k$, explicitly

$$\tilde{\Pi}_0 = 1, \qquad \tilde{\Pi}_{1,a} = t'_a, \qquad \tilde{\Pi}_{3,a} = -\frac{\partial}{\partial t'_a}\mathcal{F}_4, \qquad \tilde{\Pi}_4 = \mathcal{F}_4 \ . \tag{A.5}$$

As for the mid dimensional part, $k = 2$, we choose basis elements defined by intersections of the toric divisors $D_i = \{x_i = 0\}$

$$\gamma_1 = D_1 \cap D_2, \qquad \gamma_2 = D_2 \cap D_8, \qquad \gamma_3 = D_2 \cap D_6 \ .$$

with intersection form and inverse

$$\left(\tilde{\eta}\right)_{kl} = \gamma_k \cap \gamma_l = \begin{pmatrix} -10 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \qquad \left(\tilde{\eta}^{-1}\right)^{kl} = \begin{pmatrix} 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \ . \tag{A.6}$$

The leading parts of the $q = 2$ periods are then

$$\tilde{\Pi}_{2,1} = 5t'_1 t'_3 \, , \qquad \tilde{\Pi}_{2,2} = \frac{5}{2}t'^2_1 \, , \qquad \tilde{\Pi}_{2,3} = 2t'^2_2 \, . \tag{A.7}$$

The subleading terms of the exact periods $\Pi_{q,\cdot}$, correcting the leading parts $\tilde{\Pi}_{q,\cdot}$ away from the large $\mathrm{Im}\, t$ limit, come in two varieties. Firstly polynomial corrections in the t_a of lower degree, which involve the topological invariants (A.3).[29] For simplicity, we ignore these terms in the following, as they can be interesting for other applications[30] but do not matter for the present purpose.

Secondly, there are instanton corrections $\sim q_a = \exp 2\pi i t_a$, which will be important in the match between the open string superpotential (7.8) and the F-theory superpotential (7.9) claimed in sect. 7. These instanton corrections can be computed by mirror symmetry of the four-folds (X_c, X_c^*) as in [70,68,76] which can be summarized as follows. To determine the instanton corrections, we may simply replace the leading periods $\tilde{\Pi}_q$ in eqs. (A.5),(A.7) with the exact solutions to the Picard-Fuchs equations in eq.(4.4) with the

[29] The coefficients should be determined by a computation of the anomalous charges of wrapped D-branes, similarly as in [78].

[30] See e.g. [79].

same leading part when rewritten in t_a coordinates, using the mirror map (4.7).[31] The result for the exact periods $\Pi_{q,k} = \tilde{\Pi}_{q,k} + \mathcal{O}(q_a)$ is of the form

$$\begin{aligned}
&\Pi_0 = 1, \\
&\Pi_{1,1} = t_1' \ , \qquad \Pi_{1,2} = t_2' \ , \qquad \Pi_{1,3} = \boldsymbol{S} \cdot 1 \ , \\
&\Pi_{2,1} = \boldsymbol{S} \cdot \ 5t_1' + \pi_{2,1}, \qquad \Pi_{2,2} = -\tilde{F}_t, \qquad \Pi_{2,3} = -\tilde{W}, \\
&\Pi_{3,1} = \boldsymbol{S} \cdot \ \tilde{F}_t + \pi_{3,1} \ , \qquad \Pi_{3,2} = \tilde{T} \ , \qquad \Pi_{3,3} = -\tilde{F}_0 \ , \\
&\Pi_4 = \boldsymbol{S} \cdot \ \tilde{F}_0 + \pi_4 \ ,
\end{aligned} \tag{A.8}$$

where we have again indicated the fibration structure by explicitly indicating powers in the base volume $S = t_3 = t_3'$. The other functions $\pi_{q,k}$ denote subleading corrections whose precise form does not matter for the moment.

The leading terms in weak coupling limit $\mathrm{Im}\, S \to \infty$ are characterized by the Kähler moduli $t, \hat{t}$ and the four functions $\tilde{F}_t$, $\tilde{F}_0$ and $\tilde{W}$, $\tilde{T}$. The first two functions are equal to the periods $F_t = \partial_t \mathcal{F}(t)$ and $F_0 = 2\mathcal{F}(t) - t\partial_t \mathcal{F}(t)$ of the quintic three-fold up to instanton corrections in $q_S = \exp(2\pi i S)$, where $\mathcal{F}(t)$ denotes the exact instanton corrected prepotential of the quintic

$$\tilde{F}_t = F_t(t) + \mathcal{O}(q_S) = -\frac{5}{2}t^2 + ..., \qquad \tilde{F}_0 = F_0(t) + \mathcal{O}(q_S) = \frac{5}{6}t^3 + ... \ .$$

Similarly, the remaining functions $\tilde{W}$, $\tilde{T}$ agree with the domain wall tension and the top period computed in sects. 3-5, up to instanton corrections in S

$$\mathcal{W} = W + \mathcal{O}(q_S) = -2t_2'^2 + ... \ , \qquad \tilde{T} = T + \mathcal{O}(q_S) = \frac{2}{3}t_2'^3 + ... \ .$$

Indeed, the function W is of the form

$$W = -2t_1^2 + \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\sum_{n_a \geq 0} \frac{1}{k^2}\, N_{n_1,n_2}(q_1^{n_1} q_2^{n_2})^k \ ,$$

where the instanton numbers N_{n_1,n_2} and the classical term are exactly the same as in the superpotential $\mathcal{T}^{(1)}$ in eq.(5.3) computed for the brane compactification (Z, L) (see table. 1). This remarkable correspondence between disc instantons in a three-fold Z and sphere instantons of a dual four-fold X [8] is possible due to the coincidence of the multicover factor $\sim k^{-2}$ for discs in a three-fold [5] and spheres in a four-fold [70,68].

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[31] A more formal account of this relation between A- and B-models is elaborated on in [68,76], using Frobenius method as in [80].

From the above it follows that the exact instanton corrected four-fold periods on X_c can be informally written as[32]

$$\underline{\Pi}_\Sigma(X_c^*) = \begin{cases} (1,S)\times\Pi_\Sigma(Z^*) \\ \\ W(t,\hat{t}) \end{cases} + \dots$$

up to subleading corrections in S denoted by the dots. Thus the closed-string superpotential of the four-fold, eq.(7.9), agrees with the open-closed superpotential computed for the brane compactification (Z^*, E) in eq. (7.8), up to subleading corrections in S:

$$W(z,\hat{z},S) = \sum_{\gamma_\Sigma\in H_3(Z^*)} (N_\Sigma + S\ M_\Sigma)\int_{\gamma_\Sigma}\underline{\Omega}^{(3,0)} + \sum_{\substack{\gamma_\Sigma\in H_3(Z^*,D)\\ \partial\gamma_\Sigma\neq 0}} \hat{N}_\Sigma\int_{\gamma_\Sigma}\underline{\Omega}^{(3,0)} + \dots \tag{A.9}$$

The first term includes the three-fold flux superpotential on Z^* from both RR and NS fluxes, in agreement with our identifcation of the modulus S with the type IIB string coupling.[33] The last term describes the disc instanton corrected domain wall tension. This concludes the derivation of our claims in sect. 7 concerning the superpotential.

Eventually we can obtain the Kähler potential from (7.14), using the topological metric

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1\!\!1_3 & 0 \\ 0 & 0 & \tilde{\eta}^{-1} & 0 & 0 \\ 0 & 1\!\!1_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

It is not very illuminating to display the exact result and we restrict to discuss the leading order in an expansion for large S and near large volume. Inserting the periods (A.8) into (7.14), and keeping only the leading order terms one obtains the result in (7.15). It is straightforward to check, that this can be rewritten, to leading orders in S, in the form (7.11) with a pairing matrix $(\underline{\eta})$ of the form (7.13) with blocks

$$\eta_{Z^*} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{\eta}_{D^\sharp} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

To leading order the flat relative periods are given by

$$\underline{\tilde{\Pi}} = \big(1, t, \tilde{F}_t, \tilde{F}_0, t_1, \tilde{W}, \tilde{T}\big) = \big(1, t, -\frac{5}{2}t^2, \frac{5}{6}t^3, t_1, -2t_1^2, -\frac{2}{3}t_1^3\big) .$$

This substantiates our claim in sect. 7 concerning the Kähler potential.

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32 A similar relation holds in a more general form for any three-fold fibration over $\mathbf{P}^1$, see sect. 2 of [68].

33 This has already been observed earlier in a related context in ref. [81].

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